Quadratic Equations: completing the square

Quadratic equation has a standard form of $ax^2+bx+c=0$

- in the previous class we solved them using *factoring* and *quadratic formula*
- today we will see how to solve them by completing the square

Quadratic function looks like: $f(x)=ax^2+bx+c$

$$y=ax^2+bx+c$$

Sometimes we need it in a *complete square form*:

$f(x)=a(x-h)^2+k$ or $y=a(x-h)^2+k$

$ax^2+bx+c=0$

Quadratic Equations: completing the square

Example: let's solve the equation $12x^2-36x+12=0$ using complete square form $(x\pm m)^2=n$

Quadratic Equations: completing the square

Example: let's solve the equation $12x^2-36x+12=0$ using complete square form $(x \pm m)^2 = n$

<u>Solution</u>:

1) factor out a: $12x^2-36x+12=0$ $\longrightarrow$

$$ax^2+bx+c=0$$

Quadratic Equations: completing the square

Example: let's solve the equation $12x^2-36x+12=0$ using complete square form $(x\pm m)^2=n$

Solution:

1) factor out a: $12x^2-36x+12=0$ $\longrightarrow$ $12(x^2-3x+1)=0$

$ax^2+bx+c=0$

Quadratic Equations: completing the square

Example: let's solve the equation $12x^2-36x+12=0$ using complete square form $(x\pm m)^2=n$

Solution:

1) factor out a: $12x^2-36x+12=0$ $\longrightarrow$ $12(x^2-3x+1)=0$

2) divide both sides of the equation by 12 :

$ax^2+bx+c=0$

Quadratic Equations: completing the square

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3) divide the middle term coefficient by 2: $x^2-3x+1=0$

$$ax^2+bx+c=0$$

Quadratic Equations: completing the square

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3) divide the middle term coefficient by 2: $x^2-3x+1=0$

$$\frac{-3}{2}=-\frac{3}{2}=-1.5$$

$ax^2+bx+c=0$

Quadratic Equations: completing the square

Example: let's solve the equation $12x^2-36x+12=0$ using complete square form $(x\pm m)^2=n$

Solution:

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2) divide both sides of the equation by 12 : $x^2-3x+1=0$

3) divide the middle term coefficient by 2: $x^2-3x+1=0$

$$\frac{-3}{2}=-\frac{3}{2}=-1.5$$

4) move the constant c to the other side:

$x^2-3x+1=0 \longrightarrow$

$ax^2+bx+c=0$

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3) divide the middle term coefficient by 2: $x^2-3x+1=0$

$$\frac{-3}{2}=-\frac{3}{2}=-1.5$$

4) move the constant c to the other side:

$x^2-3x+1=0 \longrightarrow x^2-3x=-1$

$ax^2+bx+c=0$

Quadratic Equations: completing the square

Example: let's solve the equation $12x^2-36x+12=0$ using complete square form $(x\pm m)^2=n$

Solution:

5) add the square of the result from step 3 to both sides of the equation:

$$x^2-3x=-1$$

$$+\left(-\frac{3}{2}\right)^2 \quad +\left(-\frac{3}{2}\right)^2$$

$$x^2-3x+1=0$$

$$\frac{-3}{2}=-\frac{3}{2}=-1.5$$

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Example: let's solve the equation $12x^2-36x+12=0$ using complete square form $(x\pm m)^2=n$

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$$x^2-3x=-1$$

$$+\left(-\frac{3}{2}\right)^2 \qquad +\left(-\frac{3}{2}\right)^2$$

$$x^2-3x+\frac{9}{4}=-1+\frac{9}{4}$$

$$x^2-3x+1=0$$

$$\frac{-3}{2}=-\frac{3}{2}=-1.5$$

$$ax^2+bx+c=0$$

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$$x^2-3x+\frac{9}{4}=-1+\frac{9}{4}=-\frac{4}{4}+\frac{9}{4}=\frac{5}{9}$$

$$x^2-3x+1=0$$

$$\frac{-3}{2}=-\frac{3}{2}=-1.5$$

$ax^2+bx+c=0$

Quadratic Equations: completing the square

Example: let's solve the equation $12x^2-36x+12=0$ using complete square form $(x\pm m)^2=n$

Solution:

6) re-write the left side of equation as (x+result from step 3)^2, keep the right side as is:

$$\left(x-\frac{3}{2}\right)^2=\frac{5}{9}$$

$$x^2-3x+\frac{9}{4}=-1+\frac{9}{4}=-\frac{4}{4}+\frac{9}{4}=\frac{5}{9}$$

$$x^2-3x+1=0$$

$$\frac{-3}{2}=-\frac{3}{2}=-1.5$$

Quadratic Equations: completing the square

Example: let's solve the equation $12x^2-36x+12=0$ using complete square form $(x\pm m)^2=n$

Solution:

7) Finish solving the equation: recall the property we used before, if $x^2=k$, then $x=\pm\sqrt{k}$ (assuming that $k\geq 0$)

$$\left(x-\frac{3}{2}\right)^2=\frac{5}{9} \longrightarrow$$

$ax^2+bx+c=0$

Quadratic Equations: completing the square

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$$\left(x-\frac{3}{2}\right)^2=\frac{5}{9} \longrightarrow x-\frac{3}{2}=\pm\sqrt{\frac{5}{9}}=\pm\frac{\sqrt{5}}{3}$$

$ax^2+bx+c=0$

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$$+\frac{3}{2} \qquad\qquad +\frac{3}{2}$$

$$x=\pm\frac{\sqrt{5}}{3}+\frac{3}{2}$$

$ax^2+bx+c=0$

Quadratic Equations: completing the square

Example: let's solve the equation $12x^2-36x+12=0$ using complete square form $(x\pm m)^2=n$

Answer: $\left\{-\frac{\sqrt{5}}{3}+\frac{3}{2},\frac{\sqrt{5}}{3}+\frac{3}{2}\right\}$

$ax^2+bx+c=0$

Quadratic Equations: completing the square

Example: let's re-write the expression $12x^2-36x+12$ in a complete square form $a(x-h)^2+k$

$a x^2 + b x + c = 0$

Quadratic Equations: completing the square

Example: let's re-write the expression $12x^2 - 36x + 12$
in a complete square form $a(x-h)^2 + k$

Solution:
1) factor out a: $12x^2 - 36x + 12$

$ax^2+bx+c=0$

Quadratic Equations: completing the square

Example: let's re-write the expression $12x^2-36x+12$ in a complete square form $a(x-h)^2+k$

Solution:

1) factor out a: $12x^2-36x+12 \longrightarrow 12(x^2-3x+1)$

$$ax^2+bx+c=0$$

Quadratic Equations: completing the square

Example: let's re-write the expression $12x^2-36x+12$
in a complete square form $a(x-h)^2+k$

<u>Solution</u>:
1) factor out a: $12x^2-36x+12 \longrightarrow 12(x^2-3x+1)$

2) divide the middle term coefficient by 2:

$$x^2-3x+1=0$$

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3) compose the square:

(x + result from step 2)^2, open parentheses:

$ax^2+bx+c=0$

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$$\frac{-3}{2}=-\frac{3}{2}=-1.5$$

3) compose the square: (x + result from step 2)^2, open parentheses: $\left(x-\frac{3}{2}\right)^2=x^2-3x+\frac{9}{4}$

Quadratic Equations: completing the square

Example: let's re-write the expression $12x^2-36x+12$ in a complete square form $a(x-h)^2+k$

Solution:

3) So far we have:

$$12x^2-36x+12=12(x^2-3x+1)$$

and we know that

$$\left(x-\frac{3}{2}\right)^2=x^2-3x+\frac{9}{4}$$

$$ax^2+bx+c=0$$

Quadratic Equations: completing the square

Example: let’s re-write the expression $12x^2-36x+12$ in a complete square form $a(x-h)^2+k$

Solution:

3) So far we have:

$$12x^2-36x+12=12(x^2-3x+1)=12\left(x-\frac{3}{2}\right)^2+?$$

and we know that

$$\left(x-\frac{3}{2}\right)^2=x^2-3x+\frac{9}{4}$$

$$?=$$

$$ax^2+bx+c=0$$

Quadratic Equations: completing the square

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and we know that

$$\left(x-\frac{3}{2}\right)^2=x^2-3x+\frac{9}{4}$$

$$?=12\left(1-\frac{9}{4}\right)$$

$$ax^2+bx+c=0$$

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Example: let's re-write the expression $12x^2-36x+12$ in a complete square form $a(x-h)^2+k$

Solution:

3) So far we have:

$$12x^2-36x+12=12(x^2-3x+1)=12\left(x-\frac{3}{2}\right)^2+?=12\left(x-\frac{3}{2}\right)^2-15$$

and we know that

$$\left(x-\frac{3}{2}\right)^2=x^2-3x+\frac{9}{4}$$

$$?=12\left(1-\frac{9}{4}\right)=12\left(-\frac{5}{4}\right)=-15$$

Quadratic Equations: completing the square

Example: let's re-write the expression $12x^2-36x+12$ in a complete square form $a(x-h)^2+k$

<u>Answer</u>: $12x^2-36x+12=12\left(x-\frac{3}{2}\right)^2-15$

Quadratic Equations: completing the square

Exercises:

(1) solve the equation $2x^2 - 12x - 14 = 0$ by completing the square

(2) re-write the expression $2x^2 - 12x - 14$ in the form

$$a(x-h)^2 + k$$

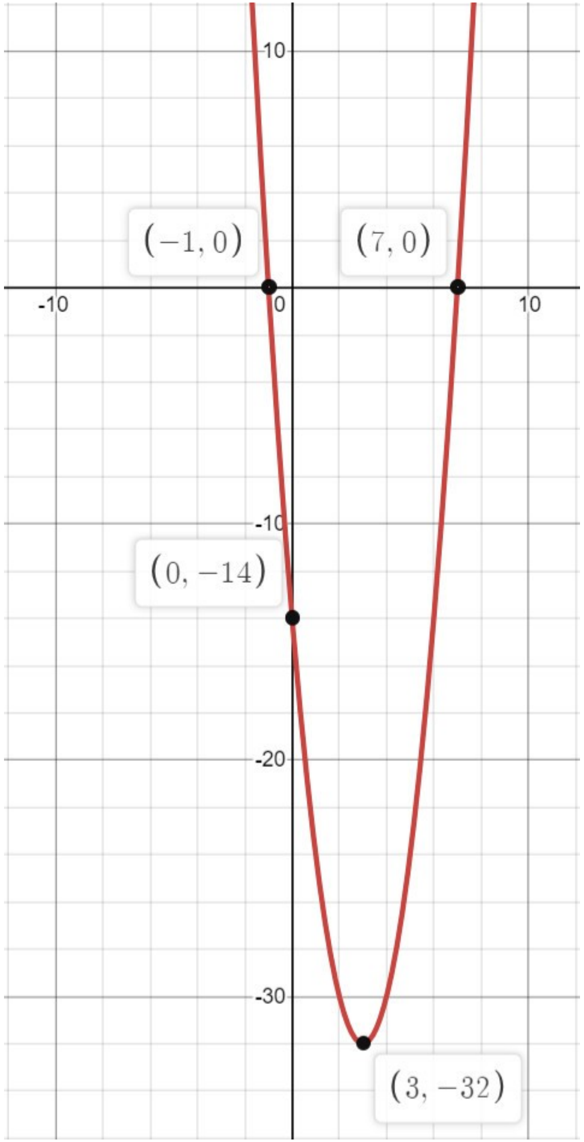
Graphs of quadratic functions

Function $f(x)=ax^2+bx+c$ is called *quadratic function* and has a graph called *parabola*.

Example: let's graph function

$$f(x)=2x^2-12x-14$$

using Desmos graphing calculator:

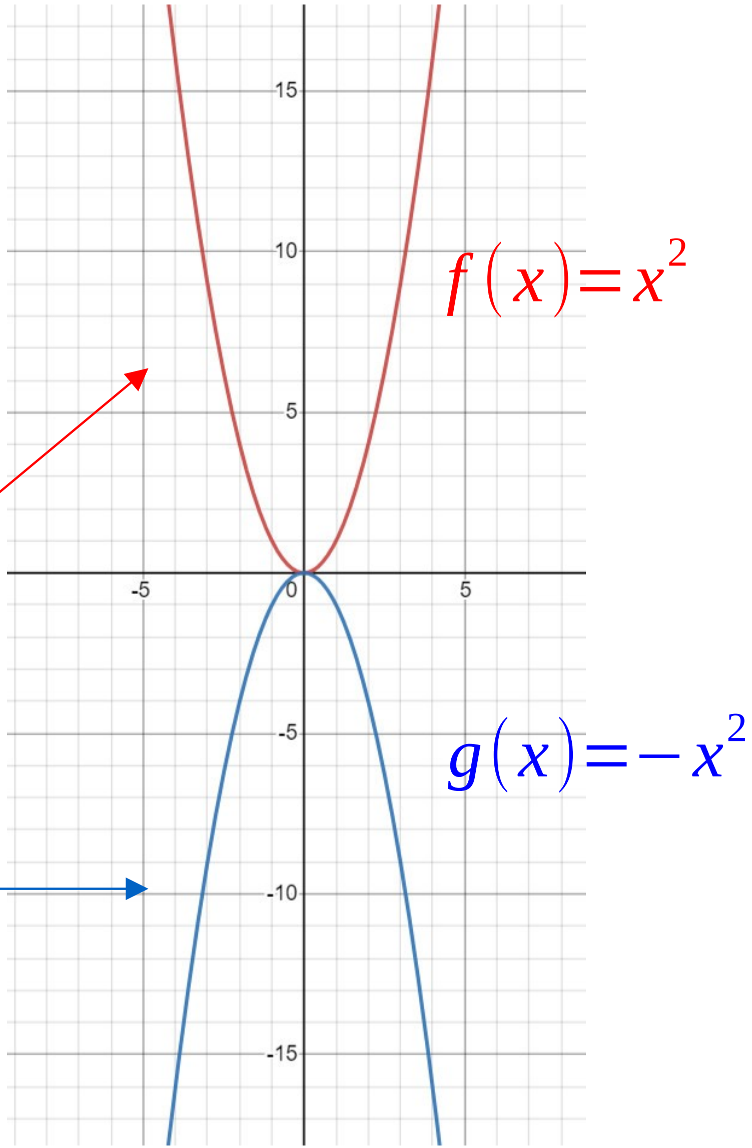


Graphs of quadratic functions

Parabola can be *cupped upwards* or downwards:

$$f(x) = a x^2 + b x + c$$

- if a > 0, then the parabola is cupped upwards
- if a < 0, then the parabola is cupped downwards

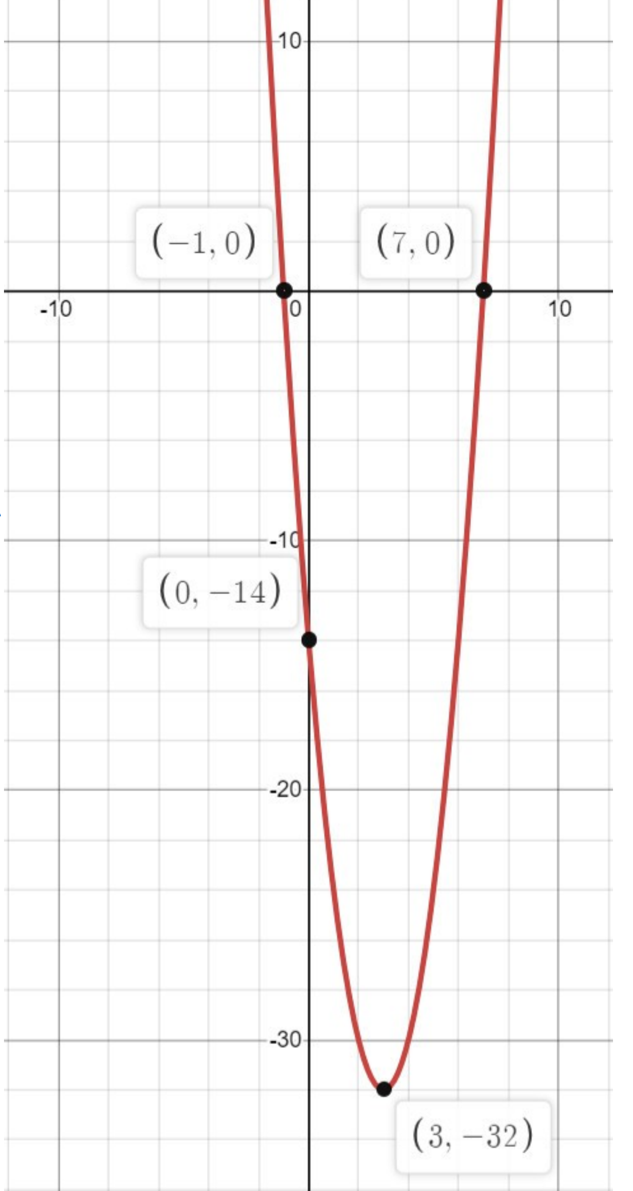


Graphs of quadratic functions

Parabola has *vertical axis of symmetry*, that passes through the vertex.

Example: let's find the equation for the *axis of symmetry* for the graph of

$$f(x) = 2x^2 - 12x - 14$$



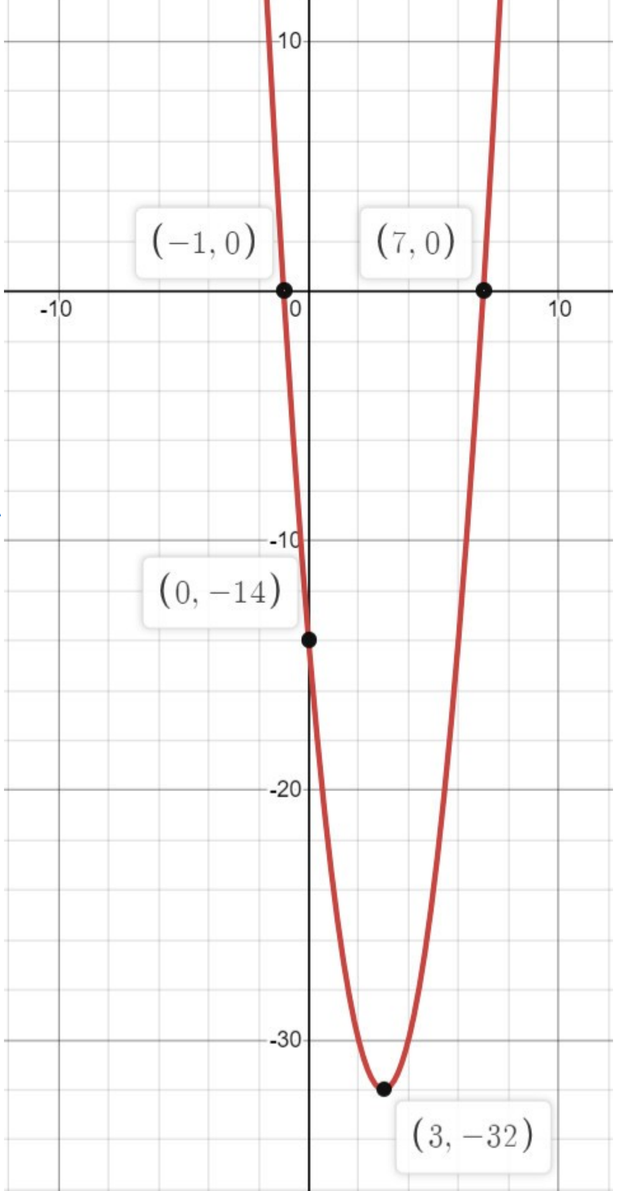
Graphs of quadratic functions

Parabola has *vertical axis of symmetry*, that passes through the vertex.

Example: let's find the equation for the *axis of symmetry* for the graph of

$$f(x) = 2x^2 - 12x - 14$$

$$x = \frac{-b}{2a}$$



Graphs of quadratic functions

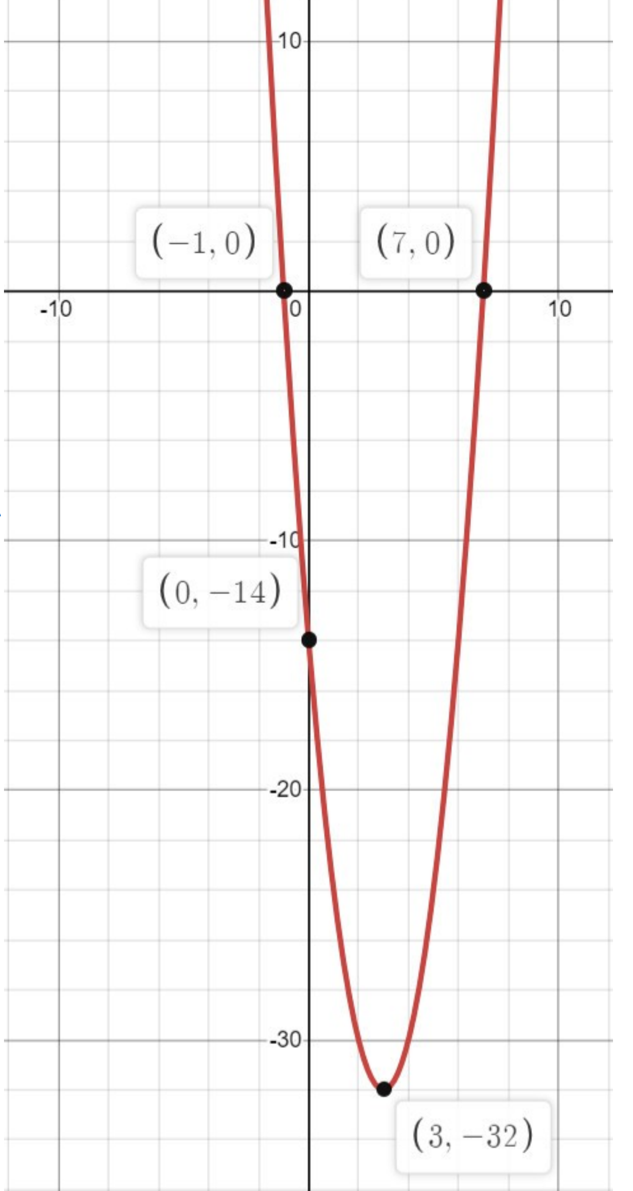
Parabola has *vertical axis of symmetry*, that passes through the vertex.

Example: let's find the equation for the *axis of symmetry* for the graph of

$$f(x) = 2x^2 - 12x - 14$$

$$x = \frac{-b}{2a} = \frac{-(-12)}{(2 \cdot 2)} = 3$$

Hence $x = 3$ is the equation for the axis of symmetry of the graph of $f(x)$.



Graphs of quadratic functions

How can we find the *vertex* of the parabola?

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We know the x-coordinate of the parabola, it is $x = \frac{-b}{2a}$

Find it's value, then use it in the equation!

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We know the x-coordinate of the parabola, it is $x = \frac{-b}{2a}$

Find it’s value, then use it in the equation!

Example: let’s find the vertex of the graph of $f(x) = -3x^2 - 4x + 1$

Graphs of quadratic functions

How can we find the *x-intercepts* and *y-intercept*, if there are any?

Graphs of quadratic functions

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- To find x-intercepts, set y = 0, find x
- To find y-intercept, set x = 0, find y

Graphs of quadratic functions

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- To find x-intercepts, set y = 0, find x
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Example: let's find the intercepts of the graph of $f(x) = x^2 + 2x - 8$

Graphs of quadratic functions

Exercise: For the graph of $f(x)=x^2+2x-8$,

(a) state whether it is *cupped upwards* or *downwards*
(b) find its *vertex* and *axis of symmetry*
(c) find its *x-intercepts* and *y-intercept*