

N4

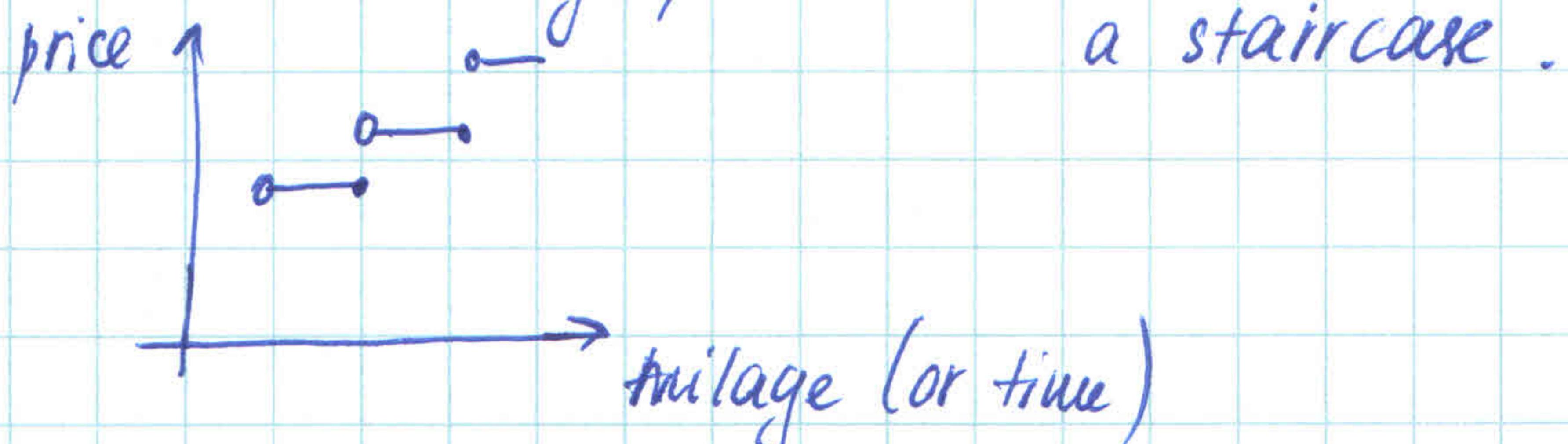
On the following intervals, function  $g$  is continuous:

$$(-4, -2), (-2, 2), (2, 4), (4, 6), (6, 8).$$

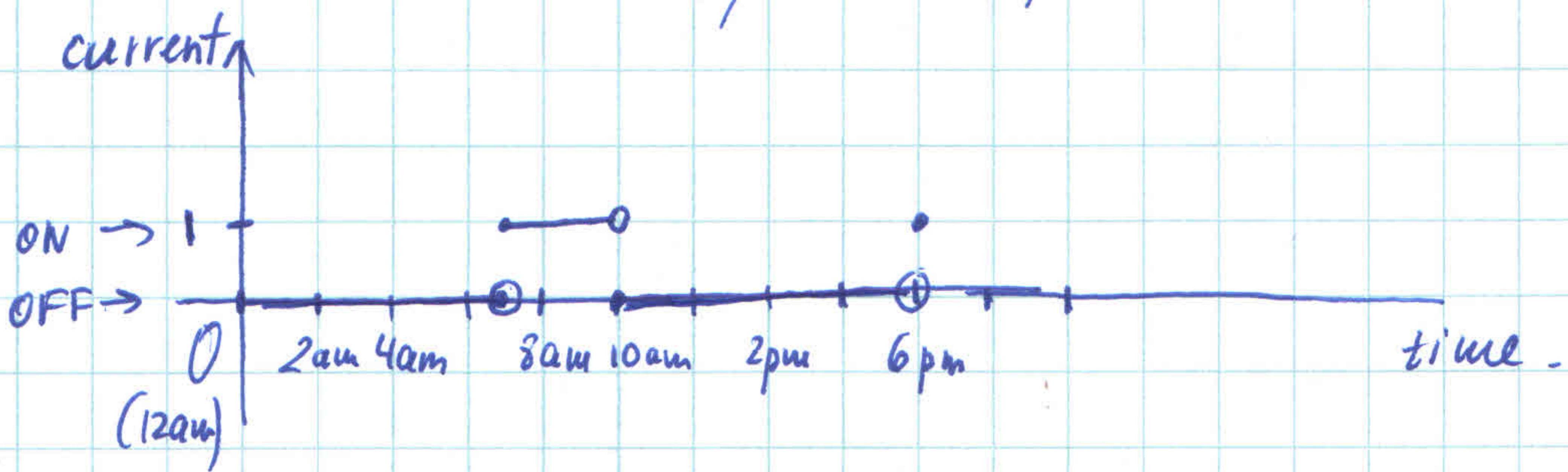
N10

a) this function is continuous, because at the same location the temperature changes gradually, there are no "jumps" in the temperature readings over a reasonably small interval of time.

d) this function is not continuous, because usually the rate is for a mile (or half a mile), so the graph of such a function will look like a staircase.



e) not a continuous function (discontinuous) - see an example of a function below (its graph)



N12

$$f(x) = 3x^4 - 5x + \sqrt[3]{x^2+4}, \quad a=2$$

Solution: let  $g(x) = 3x^4 - 5x$  and  $h(x) = \sqrt[3]{x^2+4}$

- 1) By Theorem 5,  $g(x)$  is a polynomial function, continuous on  $\mathbb{R}$  (includes 2)
- 2) By Theorem 7,  $h(x)$  root function, continuous on  $\mathbb{R}$  (includes 2)
- 3) Since  $f(x) = g(x) + h(x)$ , by Theorem 4, it is continuous on  $\mathbb{R}$  (including 2)

$$N16 \quad g(x) = 2\sqrt{3-x}, \quad (-\infty, 3]$$

Solution:  $g(x)$  is a root function. According to the theorem 7 if  $\sqrt{u}$  is continuous everywhere on its domain.

Let's find its domain:  $3-x \geq 0$ , i.e.  $x \leq 3$

The domain of  $g(x)$  is  $(-\infty, 3]$  or  $\{x \in \mathbb{R} \mid x \leq 3\}$ .

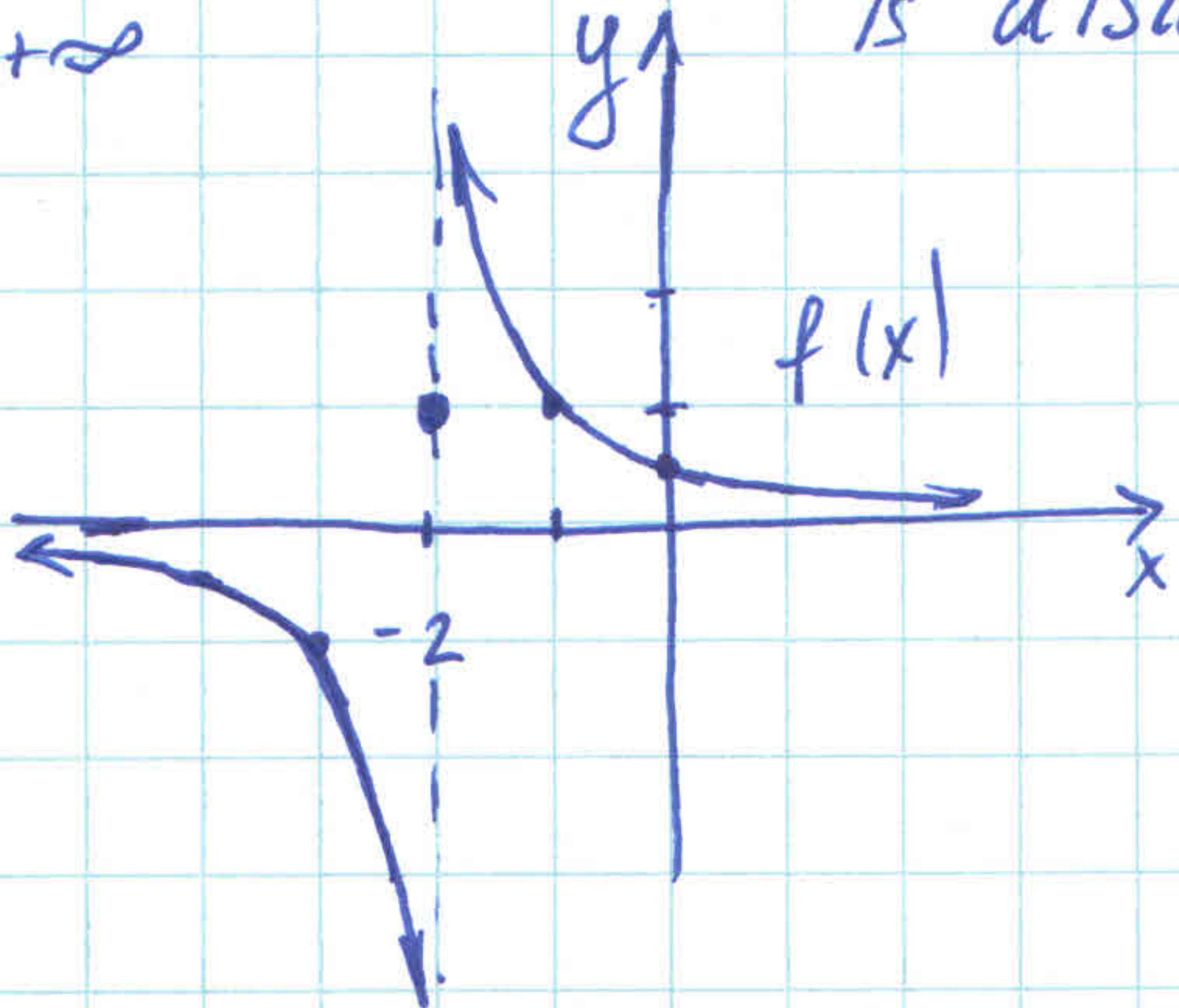
q.e.d.

$$N18 \quad f(x) = \begin{cases} \frac{1}{x+2}, & x \neq -2 \\ 1, & x = -2 \end{cases}$$

$a = -2$ , sketch graph.

Solution:  $\lim_{x \rightarrow -2^-} f(x) = -\infty \neq f(-2) = 1$ , therefore  $f(x)$  is discontinuous at  $a = -2$ .

let's sketch its graph:



$$N24 \quad f(x) = \frac{x^3 - 8}{x^2 - 4} \quad - \text{discontinuous at } 2.$$

Solution:

$$\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{(x-2)(x^2 + 2x + 4)}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{x^2 + 2x + 4}{x+2} =$$

$$= \lim_{x \rightarrow 2} \frac{4+4+4}{4} = 3$$

Hence, we can remove the discontinuity by

$$f(x) = \begin{cases} \frac{x^3 - 8}{x^2 - 4}, & x \neq 2 \\ 3, & x = 2 \end{cases}$$

N28

$$h(x) = \frac{\sin x}{x+1}$$

Solution: let  $f(x) = \sin x$ , and  $g(x) = x+1$ ,

$f(x)$  is a trigonometric function; by Theorem 7, it is continuous everywhere on its domain, which is  $\mathbb{R}$

$g(x)$  is a polynomial function; by Theorem 5 it is continuous everywhere, i.e. on  $\mathbb{R}$

$h(x) = \frac{f(x)}{g(x)}$  is continuous at any  $a$ , such that  $g(a) \neq 0$  (by Theorem 4).

Therefore, the domain of  $h(x)$  is  $\mathbb{R} - \{-1\}$  or

$\boxed{x \in \mathbb{R} \mid x \neq -1}$  or  $\boxed{(-\infty, -1) \cup (-1, \infty)}$  and it's continuous on its domain.

q.e.d.

N36

$$\lim_{x \rightarrow \pi} \sin(x + \sin x)$$

Solution: let  $g(x) = x + \sin x$  and  $f(x) = \sin x$ , then

$\sin(x + \sin x) = f(g(x)) = (f \circ g)(x)$  - composition of  $f$  and  $g$ .

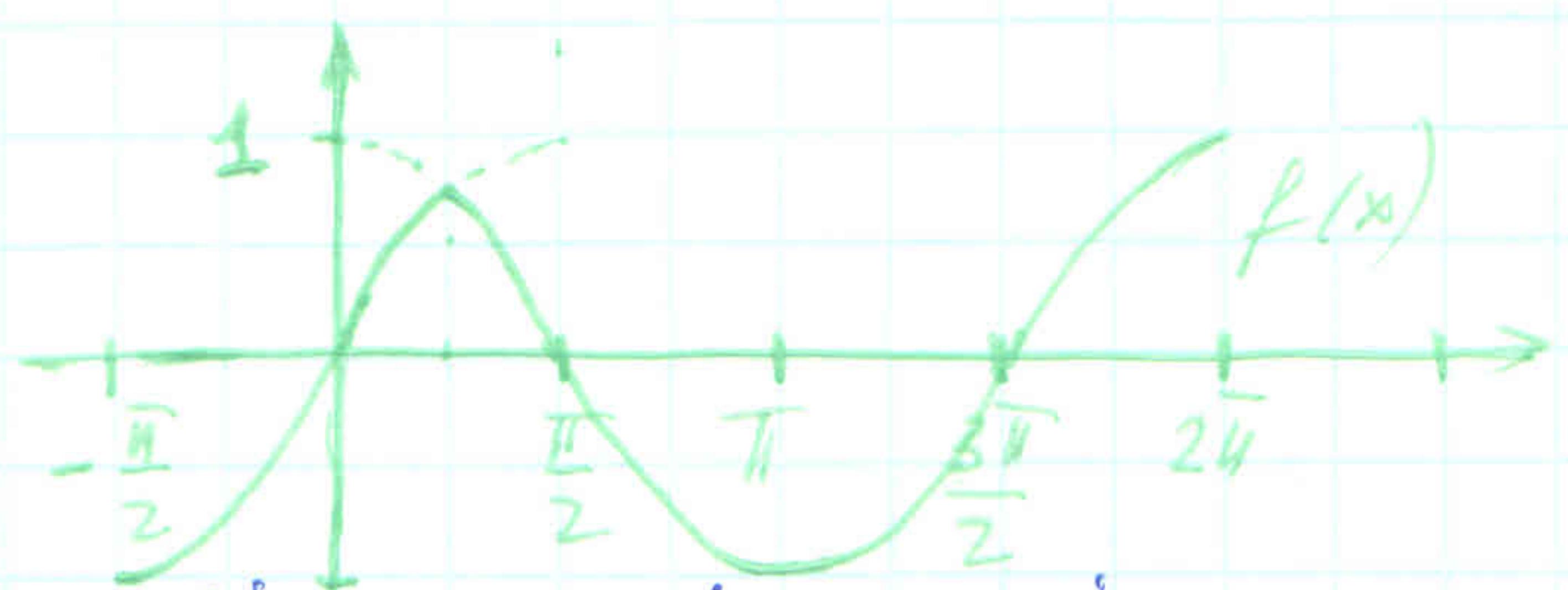
Function  $g(x)$  is continuous at  $\pi$ ,  $g(\pi) = \pi + 0 = \pi$ , and function  $f(x)$  is continuous at  $g(\pi)$  ( $f(g(\pi)) = 0$ ),

therefore  $f(g(x))$  is continuous at  $\pi$  by Theorem 7

$$\begin{aligned} \lim_{x \rightarrow \pi} \sin(x + \sin x) &= \sin \left( \lim_{x \rightarrow \pi} (x + \sin x) \right) = \sin (\pi + \sin \pi) = \\ &= \sin(\pi) = \boxed{0} \end{aligned}$$

N40

$$f(x) = \begin{cases} \sin x, & x < \frac{\pi}{4} \\ \cos x, & x \geq \frac{\pi}{4} \end{cases}$$



Solution: It is clear that  $\sin x$  and  $\cos x$  are continuous everywhere, i.e. on  $\mathbb{R}$ .

Hence,  $\sin x$  is continuous on  $(-\infty, \frac{\pi}{4})$ , and  $\cos x$  is continuous on  $(\frac{\pi}{4}, +\infty)$ .

Therefore, we need to check what happens at  $\frac{\pi}{4}$ .

$$\lim_{x \rightarrow \frac{\pi}{4}^-} \sin x = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} = f\left(\frac{\pi}{4}\right) \quad \left. \begin{array}{l} \text{hence, } f(x) \\ \text{is continuous} \\ \text{at } \frac{\pi}{4} \text{ by} \\ \text{definition.} \end{array} \right\}$$

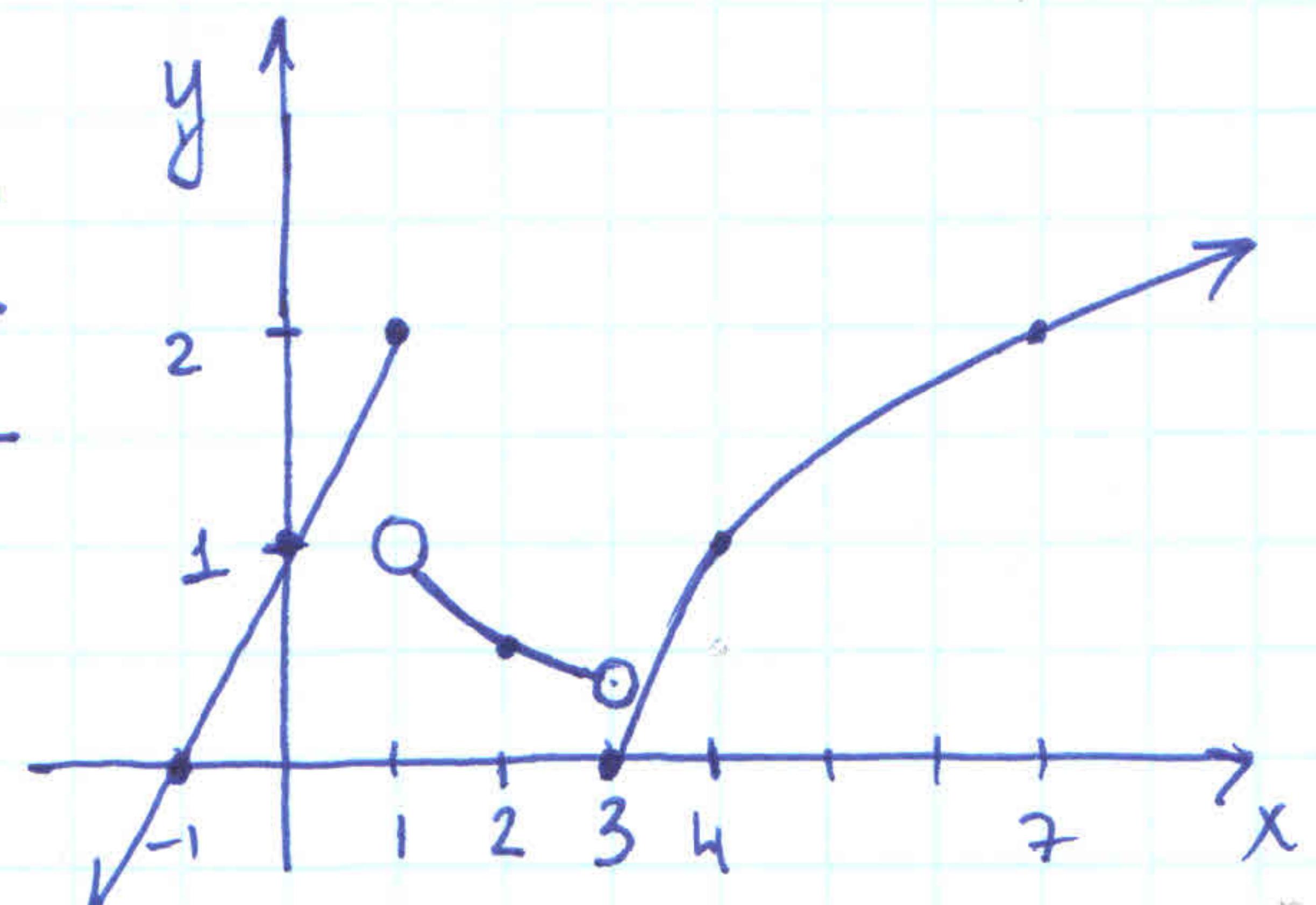
- So, we proved that  $f(x)$  is continuous on  $\mathbb{R}$ .

N42

$$f(x) = \begin{cases} x+1, & x \leq 1 \\ \frac{1}{x}, & 1 < x < 3 \\ \sqrt{x-3}, & x \geq 3 \end{cases}$$

Solution: let's sketch the graph :

x	x+1	x	$\frac{1}{x}$	$\sqrt{x-3}$
-1	0	1	1	hyperbola
1	2	2	1/2	exclude
0	1	3	1/3	exclude



By looking at graph we can conclude that  $f(x)$  is discontinuous at 1, and 3. ;

continuous from the left at 1, from the right at 3.

If we didn't sketch the graph first, then we would

check the boundary values of  $x = 1, 3$  (by looking at the function's definition)

(1)  $f(1) = 2$

$$\lim_{x \rightarrow 1^-} f(x) = 1+1 = 2 \quad , \quad \lim_{x \rightarrow 1^+} f(x) = \frac{1}{x} = 1$$

using  $\frac{1}{x+1}$                                     using  $\frac{1}{x}$

- different limits, therefore  $\lim_{x \rightarrow 1} f(x)$  doesn't exist.
- from this data we can conclude that:

- 1) function  $f(x)$  is discontinuous at 1, and
  - 2) continuous from the left at 1

(2)  $f(3) = \sqrt{3-3} = \sqrt{0} = 0$

$$\lim_{x \rightarrow 3^-} f(x) = \frac{1}{3} \neq 0 \quad , \quad \lim_{x \rightarrow 3^+} f(x) = \sqrt{3-3} = 0$$

using  $\frac{1}{x}$                                     using  $\sqrt{x-3}$

- different limits, therefore  $\lim_{x \rightarrow 3} f(x)$  doesn't exist
- from this data we can conclude that:

- 1) function  $f(x)$  is discontinuous at 3, and
  - 2) continuous from the right at 3.

N54  $\sin x = x^2 - x \quad (1,2)$  - use the Intermediate value Theorem.

Solution: let's move everything to the left side:

$$\sin x - x^2 + x = 0 . \quad \text{Let } f(x) = \sin x - x^2 + x.$$

$$\begin{aligned} f(1) &= \sin 1 - 1 + 1 = \sin 1 \approx 0.84 \\ f(2) &= \sin 2 - 4 + 2 = \sin 2 - 2 \approx -1.09 \end{aligned} \quad \text{by the Intermediate Value Theorem,}$$

there must be a value  $c \in (1,2)$ , such that  $f(c) = 0$   
( $-1.09 < f(c) < 0.84$ ).