Chapter 3 overview

3.1: *Maximum* and *minimum* values (local and absolute/global), *critical values*, etc.

3.2: *Rolle's Theorem* and the *Mean Value Theorem*

3.3 – 3.5: *Increasing/decreasing test*, *The First Derivative Test*, etc. to aid in graph sketching.

3.6: didn't cover

3.7: Optimization problems
3.8: Newton-Raphson method
3.9: Antiderivatives

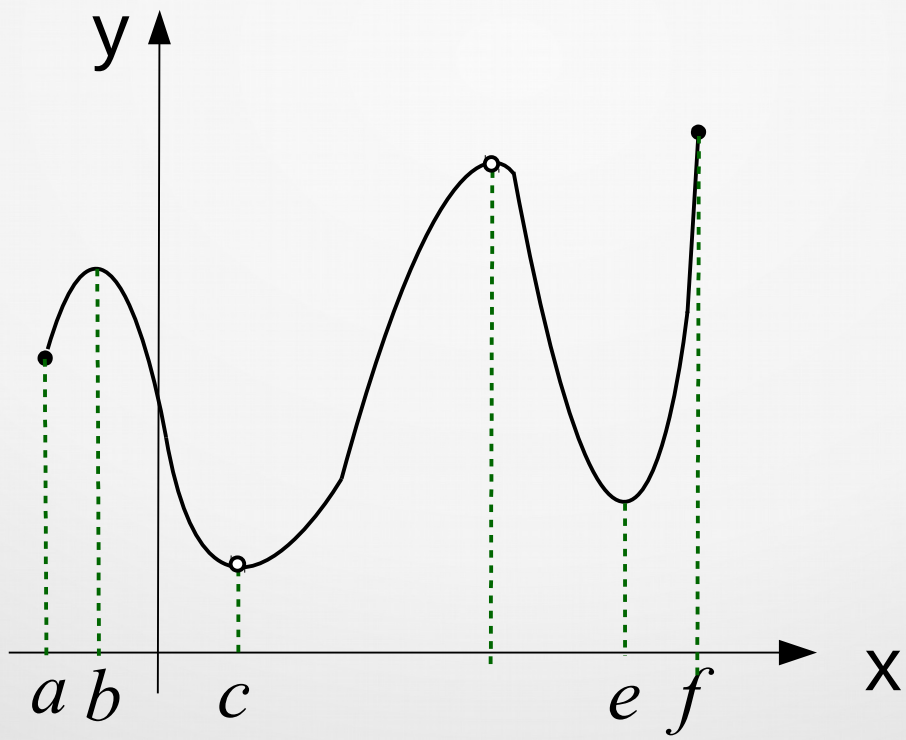
Maximum and Minimum

Let $c \in D$, domain of function f. Then $f(c)$ is

- *absolute/global maximum value* of f on D if $f(c) \geq f(x)$ for all $x \in D$
- *absolute minimum value* (*global minimum*) of f on D if $f(c) \leq f(x)$ for all $x \in D$
- *local maximum value* of f if $f(c) \geq f(x)$ true on an open interval containing c.
- *local minimum value* of f if $f(c) \leq f(x)$ true on an open interval containing c.

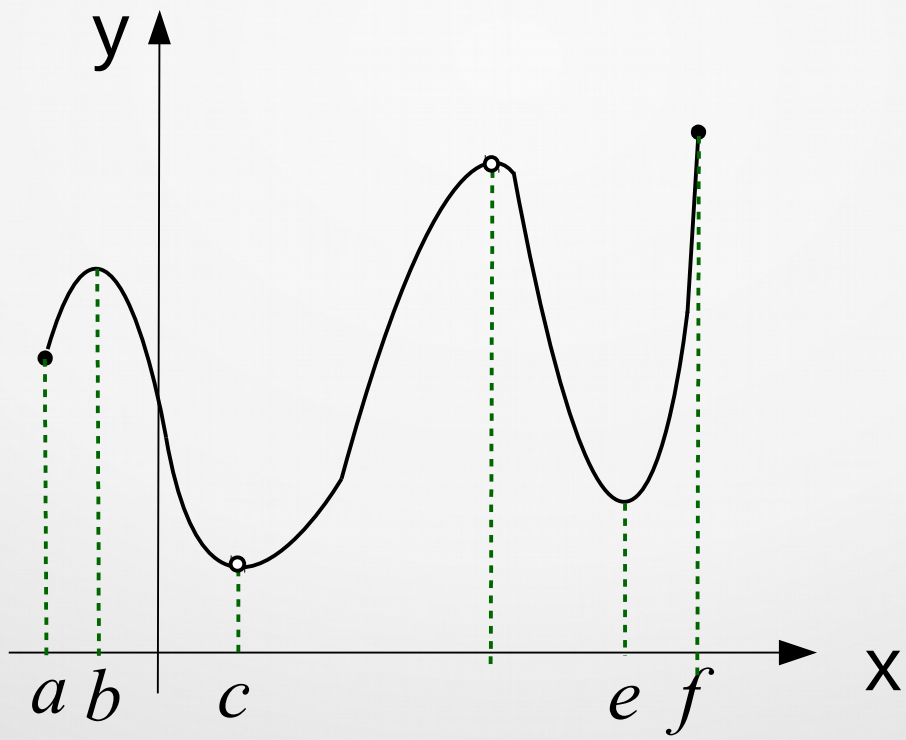
Maximum and Minimum Values

absolute maximum at:
absolute minimum at:
local maximum at:
local minimum at:



Maximum and Minimum Values

absolute maximum at: f
absolute minimum at: no absolute minimum
local maximum at: b
local minimum at: e



Maximum and Minimum Values

[Theorem] The Extreme Value Theorem

If f is a continuous on a closed interval $[a,b]$, then f attains an *absolute maximum* value $f(c)$ and an *absolute minimum* value $f(d)$ at some numbers c and d in $[a,b]$.

[Theorem] Fermat's Theorem

If f has a local maximum or minimum at c, and $f'(c)$ exists, then $f'(c)=0$ (then c is a *critical number* of f).

[Def] A *critical number* of a function f is a number c in the domain D of f such that either $f'(c)=0$ or $f'(c)$ does not exist.

! Vice-versa is not true, i.e. sometimes $f'(c)=0$, but there is no local maximum or minimum at c.

Maximum and Minimum Values

The Closed Interval Method

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the *critical numbers* of f in $[a, b]$.

2. Find the values of f at the endpoints of the interval.

3. The largest of the values from Steps 1 and 2 is the *absolute maximum value;* the smallest of these values is the *absolute minimum value*.

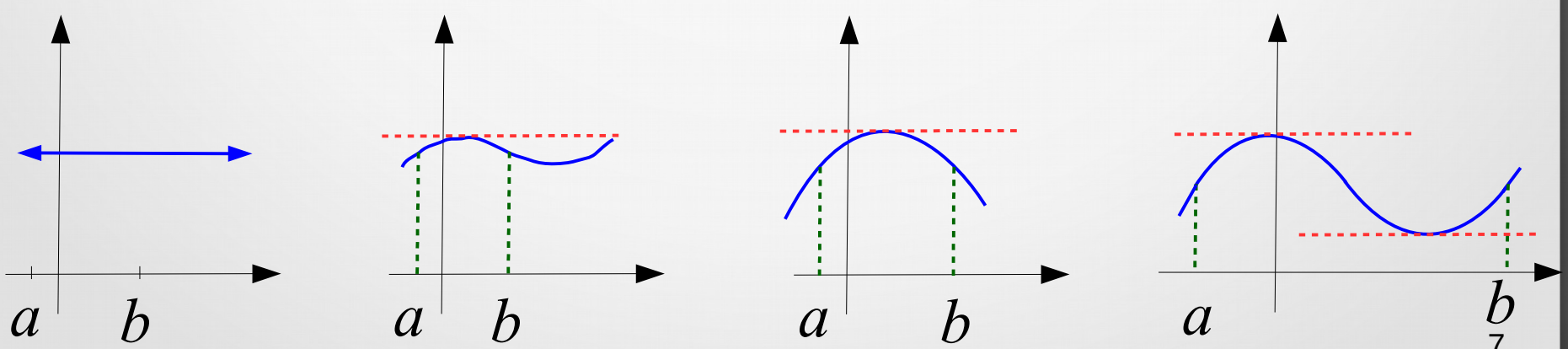
Section 3.2 The Mean Value Theorem

[Theorem] Rolle's theorem

Let f be a function that satisfies the conditions:

1) f is continuous on the closed interval $[a,b]$
2) f is differentiable on the open interval (a,b)
3) $f(a) = f(b)$

then there is a number $c \in (a,b)$, such that $f'(c) = 0$.



The Mean Value Theorem

[Theorem] The mean value theorem

Let f be a function that satisfies the conditions:

1) f is continuous on $[a,b]$

2) f is differentiable on (a,b)

then there is a number $c \in (a,b)$, such that

$$f'(c)=\frac{f(b)-f(a)}{b-a} \quad \text{or} \quad f(b)-f(a)=f'(c)(b-a)$$

this theorem allows us to obtain the information about a function from information about its derivative

The Mean Value Theorem

[Theorem]

If $f'(x) = 0$ for all x in the interval $[a,b]$, then f is constant on (a,b).

[Corollary]

If $f'(x) = g'(x)$ for all x in the interval (a,b), then $f - g$ is constant on (a,b), that is $f(x) = g(x) + c$, where c is a constant.

The Mean Value Theorem

Example:

Is it possible that the function $f(x) = x^9 + x - 15$ has two real roots?

hint: check the first derivative of $f(x)$

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using the hint: $f'(x) = 9x^8 + 1$

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using the hint: $f'(x) = 9x^8+1>0$, so $f(x)$ is increasing on $\mathbf{R}$

The Mean Value Theorem

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Is it possible that the function $f(x) = x^9 + x - 15$ has two real roots?

hint: check the first derivative of $f(x)$

Rolle's theorem

Let f be a function that satisfies the conditions:

1) f is continuous on the closed interval $[a,b]$

2) f is differentiable on the open interval (a,b)

3) $f(a) = f(b)$ No such a and b can be found!

then there is a number $c \in (a,b)$, such that $f'(c) = 0$.

using the hint: $f'(x) = 9x^8+1>0$, so $f(x)$ *is increasing on* $\mathbf{R}$

The Mean Value Theorem

Example 1:

For $f(x) = x^3 - x^2 - 6x + 2$, verify that $f(x)$ satisfies all three hypotheses of the Rolle's theorem on the interval $[0,3]$. Then find all critical numbers c that satisfy the conclusion of the Rolle's theorem.

The Mean Value Theorem

Example 2: For $f(x)=\frac{1}{x}$, verify that $f(x)$ satisfies the hypotheses of the Mean Value Theorem on the interval $[1,3]$. Then find all numbers c that satisfy the conclusion of the theorem.

The Mean Value Theorem

Example 3:

Suppose $f(x)$ is continuous on $[4,8]$ and $-2 \leq f'(x) \leq 2$ for all x in $(4,8)$. Use the *Mean Value Theorem* to estimate $f(8)-f(4)$.

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$f'(x) = 3x^2 - 2x - 6$

$3x^2 - 2x - 6 = 0$

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$f'(x) = 3x^2 - 2x - 6$

$3x^2 - 2x - 6 = 0$

$ax^2 + bx + c = 0$

Let's use quadratic formula:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 3 \cdot (-6)}}{2 \cdot 3} = \frac{1 \pm \sqrt{19}}{3}$$

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Note that $\frac{1-\sqrt{19}}{3} \notin [1,3]$, hence the answer: $\frac{1+\sqrt{19}}{3}$

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Example 2: For $f(x)=\frac{1}{x}$, verify that $f(x)$ satisfies the hypotheses of the Mean Value Theorem on the interval $[1,3]$. Then find all numbers c that satisfy the conclusion of the theorem.

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$$f'(c)=\frac{f(b)-f(a)}{b-a}=\frac{\frac{1}{3}-\frac{1}{1}}{3-1}=-\frac{1}{3}$$

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Let's find c such that $f'(c)=-\frac{1}{3}$:

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$$-\frac{1}{x^2}=-\frac{1}{3} \qquad x^2=3 \qquad x=\pm\sqrt{3}$$

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$$-\frac{1}{x^2}=-\frac{1}{3} \qquad x^2=3 \qquad x=\pm\sqrt{3}$$

$-\sqrt{3} \notin [1,3]$, hence the answer is $\sqrt{3}$

The Mean Value Theorem

Example 3:

Suppose $f(x)$ is continuous on $[4,8]$ and $-2 \le f'(x) \le 2$ for all x in $(4,8)$. Use the *Mean Value Theorem* to estimate $f(8)-f(4)$.

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Let f be a function that satisfies the conditions:

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$$f(b)-f(a)=f'(c)(b-a)$$

$$f(8)-f(4)=f'(c)(8-4)=4f'(c)$$

$$-2\cdot 4 \le f(8)-f(4)=4f'(c) \le 2\cdot 4$$

$$-8 \le f(8)-f(4) \le 8$$

The Mean Value Theorem

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$$-2\cdot 4 \le f(8)-f(4)=4f'(c) \le 2\cdot 4$$

$$-8 \le f(8)-f(4) \le 8$$

Answer: $-8 \le f(8)-f(4) \le 8$

Sections 3.3 – 3.5

Increasing / Decreasing Test

(a) if $f'(x) > 0$ on an interval,
then $f(x)$ is *increasing* on that interval.

(b) if $f'(x) < 0$ on an interval,
then $f(x)$ is *decreasing* on that interval.

Local Extreme Values: The First Derivative Test

If c is a *critical number* of a *continuous* function f. Then

(a) if f' changes from + to – at c,
then f has a *local maximum* at c.

(b) if f' changes from – to + at c,
then f has a *local minimum* at c.

(c) if f' does not change sign at c,
then f has no *local maximum* or *local minimum* at c.

Sections 3.3 – 3.5

Concavity Test

(a) if $f'' > 0$ for all x in interval I,
then the graph of f is *concave upward* on I.

(b) if $f'' < 0$ for all x in interval I,
then the graph of f is *concave downward* on I.

inflection points : f is continuous there and the curve changes from concave upward to concave downward, or vice versa.

The Second Derivative Test

If $f''(x)$ is continuous near c, and c is a critical value.

(a) if $f'(c) = 0$ and $f''(c) > 0$ then f has a *local min.* at c.

(b) if $f'(c) = 0$ and $f''(c) < 0$ then f has a *local max.* at c.

Section 3.3 – 3.5

Horizontal and vertical asymptotes

- the line $y = L$ is a *horizontal asymptote* of the curve

$y = f(x)$ if either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$

- the line $x = a$ is a *vertical asymptote* of the curve

$y = f(x)$ if either $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$

the possible values of a: $f(a)$ is undefined because of the division by zero

Section 3.3 – 3.5

Asymptotes of rational functions

$$f(x)=\frac{ax^n+\ldots}{bx^m+\ldots}$$

n is the largest exponent in the numerator and m is the largest exponent in the denominator.

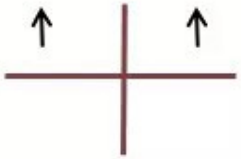
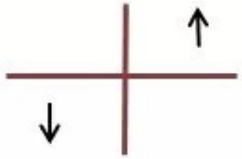
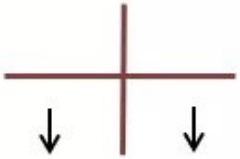
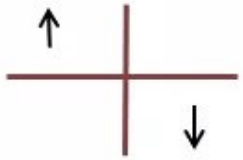
- if the denominator is zero at $x = c$ and the numerator isn't zero at $x = c$, then $x = c$ is the vertical asymptote
- if $n < m$ then the x -axis is the horizontal asymptote
- if $n = m$ then the line $y = a / b$ is the horizontal asymptote.
- if $n > m$ there will be no horizontal asymptotes.

the *slant asymptote* is the quotient of the division

Sections 3.3 – 3.5

End behavior of polynomial functions

Let's recall it:

Degree → / Sign of Leading Coefficient ↓	Even	Odd
	Examples: $-x^4+3x^2+1$, $-(x+3)^3(x+2)$, $x(3-x)^3(x+2)^2$	Examples: x^5+3x^2+1, $(x+3)^3(x+2)^2$, $x(3-x)^3(x+2)$
Positive (+) Examples: x^4+3x^2+1, $(x+3)^3(x+2)$, $(3-x)^4(x+2)$	 $x\to-\infty,\ y\to\infty$ $x\to\infty,\ y\to\infty$ Example: $y=x^2$	 $x\to-\infty,\ y\to-\infty$ $x\to\infty,\ y\to\infty$ Examples: $y=x,\ y=x^3$
Negative (–) Examples: $-x^4+3x^2+1$, $-(x+3)^3(x+2)$, $(3-x)^3(x+2)$	 $x\to-\infty,\ y\to-\infty$ $x\to\infty,\ y\to-\infty$ Example: $y=-x^2$	 $x\to-\infty,\ y\to\infty$ $x\to\infty,\ y\to-\infty$ Examples: $y=-x,\ y=-x^3$

Sections 3.3 – 3.5

[Theorem]

If r > 0 is a rational number, then $\lim_{x \to \infty} \frac{1}{x^r} = 0$

If r > 0 is a rational number such that x^r is defined for all x, then $\lim_{x \to -\infty} \frac{1}{x^r} = 0$

In addition, most of the limit laws from the Section 1.6 (except for #9 and #10) will hold if $\mathbf{a}$ is replaced by ∞.

Useful limits from Section 2.4:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad f(x) = \frac{\cos x - 1}{x}$$

Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

1. Find the domain
2. Find x- and y-intercepts
3. Check symmetry
4. Asymptotes
5. Intervals of increase/decrease
6. Concavity intervals and inflection points
7. Missing information (try some x-values)

Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

1. Find the domain

$1+\cos x=0$

Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

1. Find the domain

$1+\cos x=0 \qquad \cos x=-1 \qquad x=\pm\pi,\pm 3\pi,\pm 5\pi,\ldots$

domain $D=\{x\in\mathbb{R} \mid x\neq n\pi, n\in\mathbb{Z}, n \text{ is odd}\}$

Sections 3.3 – 3.5

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2. Find x- and y-intercepts:

Sections 3.3 – 3.5

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Sections 3.3 – 3.5

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Sections 3.3 – 3.5

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3. Check symmetry

$f(-x)=$

Sections 3.3 – 3.5

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$$f(0)=\frac{\sin 0}{1+\cos 0}=0$$

$(0,0)$ is x- and y- intercept

3. Check symmetry

$$f(-x)=\frac{\sin(-x)}{1+\cos(-x)}=\frac{-\sin x}{1+\cos x}=-f(x)$$

Sections 3.3 – 3.5

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2. Find x- and y-intercepts: if $x=0$, then

$$f(0)=\frac{\sin 0}{1+\cos 0}=0$$

$(0,0)$ is x- and y- intercept

3. Check symmetry

$$f(-x)=\frac{\sin(-x)}{1+\cos(-x)}=\frac{-\sin x}{1+\cos x}=-f(x)$$

odd function, x-axis symmetry

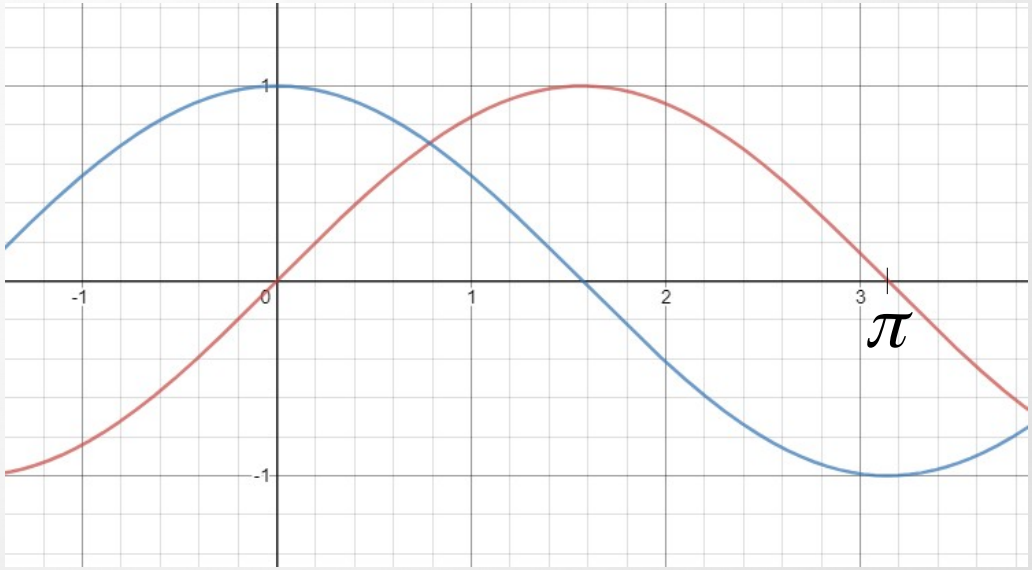
Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

4. Asymptotes

Vertical: potential places are $x=\pm\pi,\pm 3\pi,\pm 5\pi,\ldots$

$$\lim_{x\to\pi^-}\frac{\sin x}{1+\cos x}$$



Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

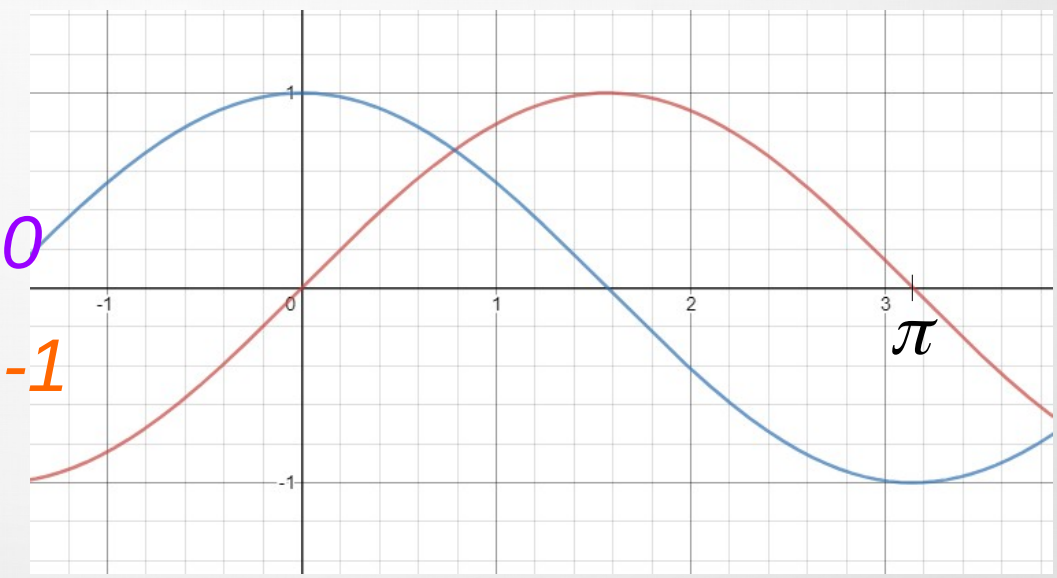
4. Asymptotes

Vertical: potential places are $x=\pm\pi,\pm 3\pi,\pm 5\pi,\ldots$

$$\lim_{x\to\pi^-}\frac{\sin x}{1+\cos x}$$

positive, approaches 0

negative, approaches -1



Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

4. Asymptotes

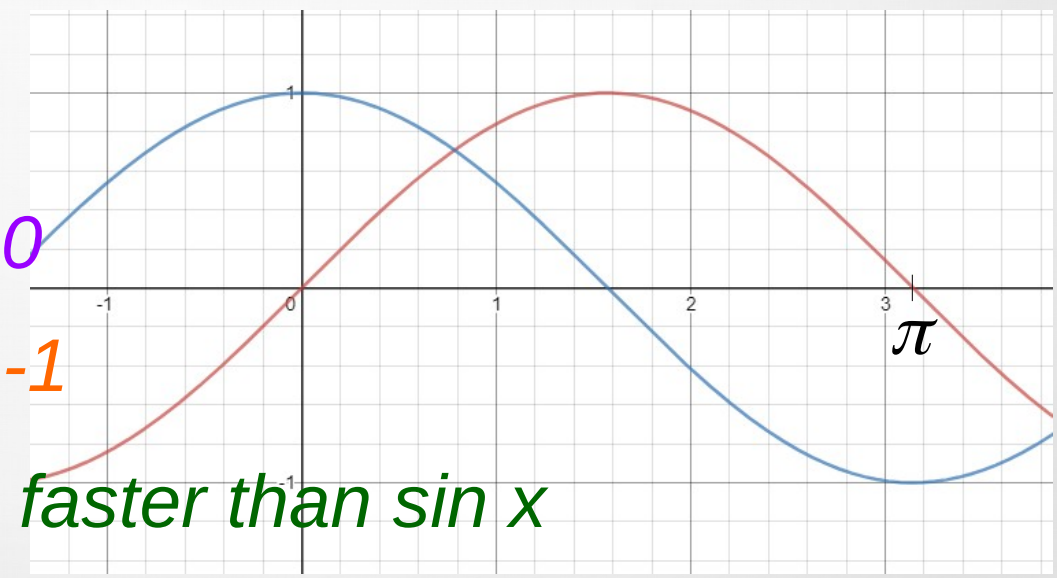
Vertical: potential places are $x=\pm\pi,\pm 3\pi,\pm 5\pi,\ldots$

$$\lim_{x\to\pi^-}\frac{\sin x}{1+\cos x}$$

positive, approaches 0

negative, approaches -1

positive, approaches 0 faster than sin x



Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

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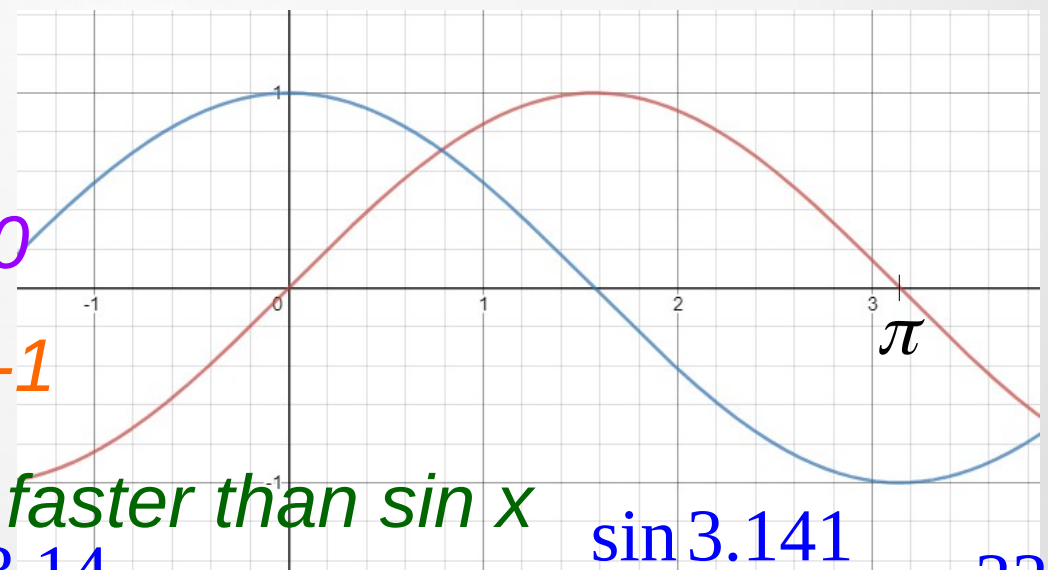


$$\frac{\sin 3.12}{1+\cos 3.12}\approx 92.6 \qquad \frac{\sin 3.14}{1+\cos 3.14}\approx 1255.8 \qquad \frac{\sin 3.141}{1+\cos 3.141}\approx 3374.7$$

Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

4. Asymptotes

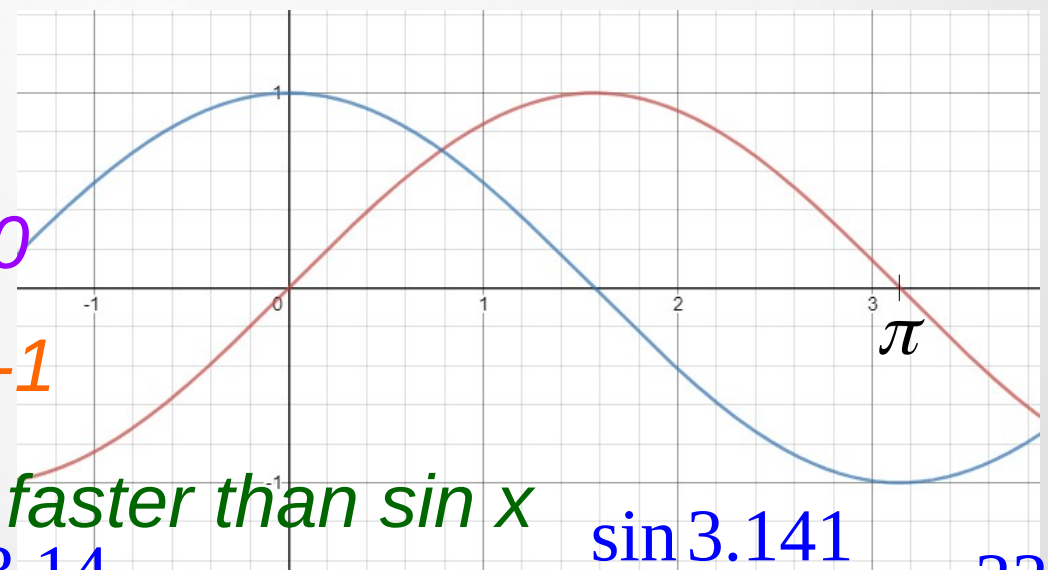
Vertical: potential places are $x=\pm\pi,\pm 3\pi,\pm 5\pi,\ldots$

$$\lim_{x\to\pi^-}\frac{\sin x}{1+\cos x}=\infty$$

positive, approaches 0

negative, approaches -1

positive, approaches 0 faster than sin x



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Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

4. Asymptotes

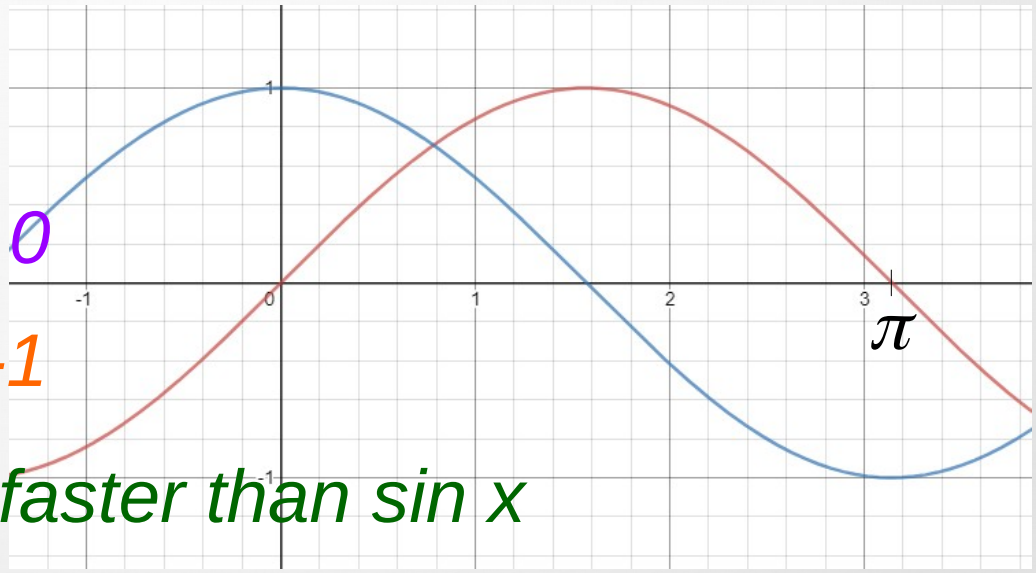
Vertical: potential places are $x=\pm\pi,\pm 3\pi,\pm 5\pi,\ldots$

$$\lim_{x\to\pi^+}\frac{\sin x}{\underbrace{1+\cos x}}=-\infty$$

negative, approaches 0

negative, approaches -1

positive, approaches 0 faster than sin x



Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

4. Asymptotes

Vertical: potential places are $x=\pm\pi,\pm 3\pi,\pm 5\pi,\ldots$

$x=\pm\pi,\pm 3\pi,\pm 5\pi,\ldots$ are vertical asymptotes

$$\lim_{x\to\pi^-}\frac{\sin x}{1+\cos x}=\infty \qquad \lim_{x\to\pi^+}\frac{\sin x}{1+\cos x}=-\infty$$

Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

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Horizontal asymptotes:

$$\lim_{x\to-\infty}\frac{\sin x}{1+\cos x}=? \qquad \lim_{x\to\infty}\frac{\sin x}{1+\cos x}=?$$

Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

4. Asymptotes

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Horizontal asymptotes:

$$\lim_{x\to-\infty}\frac{\sin x}{1+\cos x}=? \qquad \lim_{x\to\infty}\frac{\sin x}{1+\cos x}=?$$

no horizontal asymptotes

Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

5. Intervals of increase/decrease

Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

5. Intervals of increase/decrease

$$f'(x)=\frac{(\sin x)'(1+\cos x)-\sin x(1+\cos x)'}{(1+\cos x)^2}=$$

Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

5. Intervals of increase/decrease

$$f'(x)=\frac{(\sin x)'(1+\cos x)-\sin x(1+\cos x)'}{(1+\cos x)^2}=$$

$$=\frac{\cos x(1+\cos x)-\sin x(-\sin x)}{(1+\cos x)^2}=\frac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2}=$$

$$=\frac{1+\cos x}{(1+\cos x)^2}=\frac{1}{1+\cos x}$$

Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

5. Intervals of increase/decrease

$$f'(x)=\frac{1}{1+\cos x}$$

Sections 3.3 – 3.5

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$$f'(x)=\frac{1}{1+\cos x}$$

- no x-values where $f'(x)=0$
- $f'(x)$ is undefined for $x=\pm\pi,\pm 3\pi,\pm 5\pi,\ldots$

Sections 3.3 – 3.5

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Sections 3.3 – 3.5

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$$f'(0)=\frac{1}{1+\cos 0}=\frac{1}{2}>0 \qquad f'(-2\pi)=\frac{1}{1+\cos(-2\pi)}=\frac{1}{2}>0$$

$$f'(2\pi)=\frac{1}{1+\cos(2\pi)}=\frac{1}{2}>0 \qquad f'(4\pi)=\frac{1}{1+\cos(4\pi)}=\frac{1}{2}>0$$

Sections 3.3 – 3.5

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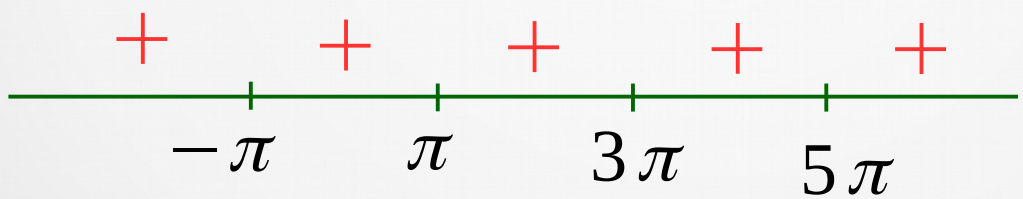

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Sections 3.3 – 3.5

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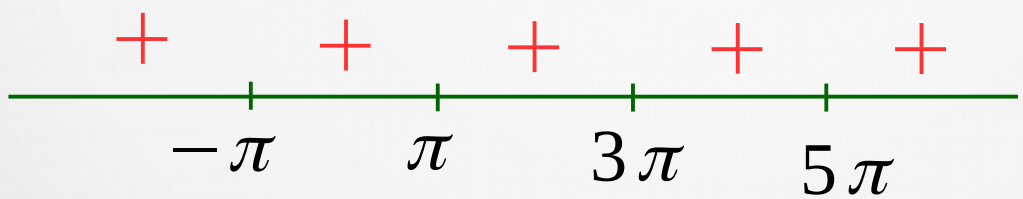

always increasing

$$f'(0)=\frac{1}{1+\cos 0}=\frac{1}{2}>0 \qquad f'(-2\pi)=\frac{1}{1+\cos(-2\pi)}=\frac{1}{2}>0$$

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Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

6. Concavity intervals and inflection points

$$f'(x)=\frac{1}{1+\cos x}$$

Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

6. Concavity intervals and inflection points

$$f'(x)=\frac{1}{1+\cos x}$$

$$f''(x)=-1\cdot(1+\cos x)^{-2}\cdot(-\sin x)$$

$$f''(x)=\frac{\sin x}{(1+\cos x)^2}$$

Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

6. Concavity intervals and inflection points

$$f''(x)=\frac{\sin x}{(1+\cos x)^2} \qquad f'(x)=\frac{1}{1+\cos x}$$

- $f(x) = 0$
- $f(x)$ is undefined

Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

6. Concavity intervals and inflection points

$$f''(x)=\frac{\sin x}{(1+\cos x)^2} \qquad f'(x)=\frac{1}{1+\cos x}$$

- $f(x) = 0$ when $x=0,\pm\pi,\pm 2\pi,\pm 3\pi,\ldots$
- $f(x)$ is undefined when $x=\pm\pi,\pm 3\pi,\pm 5\pi,\ldots$

Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

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Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

6. Concavity intervals and inflection points

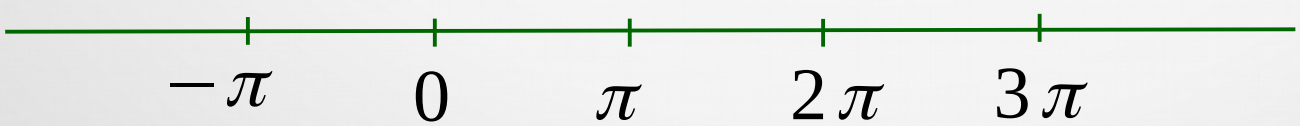
$$f''(x)=\frac{\sin x}{(1+\cos x)^2}$$

$$f'(x)=\frac{1}{1+\cos x}$$

$$f''\left(\frac{3\pi}{2}\right)=\frac{-1}{1}<0$$

- $f(x) = 0$ when $x=0,\pm\pi,\pm 2\pi,\pm 3\pi,\ldots$
- $f(x)$ is undefined when $x=\pm\pi,\pm 3\pi,\pm 5\pi,\ldots$

$$f''\left(-\frac{\pi}{2}\right)=-1<0$$



$$f''\left(\frac{\pi}{2}\right)=\frac{1}{1+\cos(\pi/2)}=\frac{1}{1}>0$$

$$f''\left(-\frac{3\pi}{2}\right)=1>0$$

Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

6. Concavity intervals and inflection points

$$f''(x)=\frac{\sin x}{(1+\cos x)^2}$$

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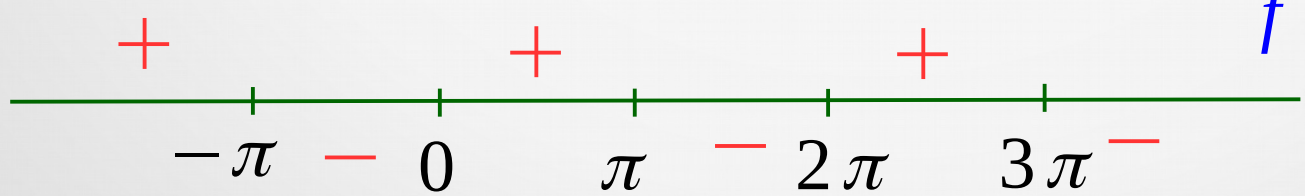

$$f''\left(\frac{\pi}{2}\right)=\frac{1}{(1+\cos(\pi/2))^2}=\frac{1}{1}>0$$

$$f''\left(-\frac{3\pi}{2}\right)=1>0$$

Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

6. Concavity intervals and inflection points

$$f''(x)=\frac{\sin x}{(1+\cos x)^2} \qquad f'(x)=\frac{1}{1+\cos x}$$

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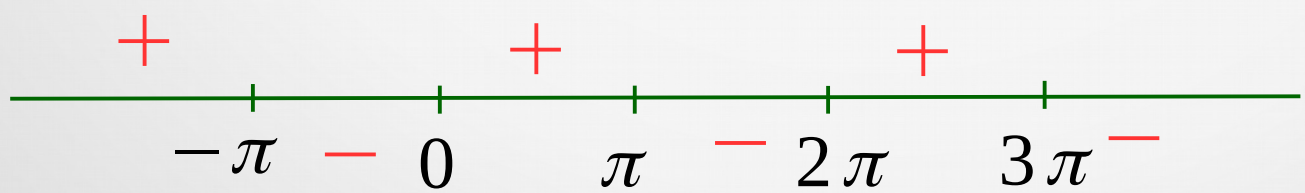

Inflection points at $x=0,\pm\pi,\pm 2\pi,\pm 3\pi,\ldots$

Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

7. Missing information (try some x-values)

$$f\left(\frac{\pi}{2}\right)=\frac{\sin(\pi/2)}{1+\cos(\pi/2)}=\frac{1}{1+0}=1$$

$$f\left(\frac{3\pi}{2}\right)=-1$$

$$f\left(-\frac{\pi}{2}\right)=\frac{-1}{1+0}=-1$$

Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

points: $\left(-\dfrac{\pi}{2},-1\right),(0,0),\left(\dfrac{\pi}{2},1\right),\left(\dfrac{3\pi}{2},-1\right)$

odd function (x-axis symmetry)

vertical asymptotes:
$x=\pm\pi,\pm 3\pi,\pm 5\pi,\ldots$

always increasing
concave up:
$\ldots,(-2\pi,-\pi),(0,\pi),(2\pi,3\pi),\ldots$
concave down:
$\ldots,(-\pi,0),(\pi,2\pi),(3\pi,4\pi),\ldots$

Sections 3.3 – 3.5

Example: Sketch the graph of $f(x)=\dfrac{\sin x}{1+\cos x}$

points: $\left(-\dfrac{\pi}{2},\right.$

odd function

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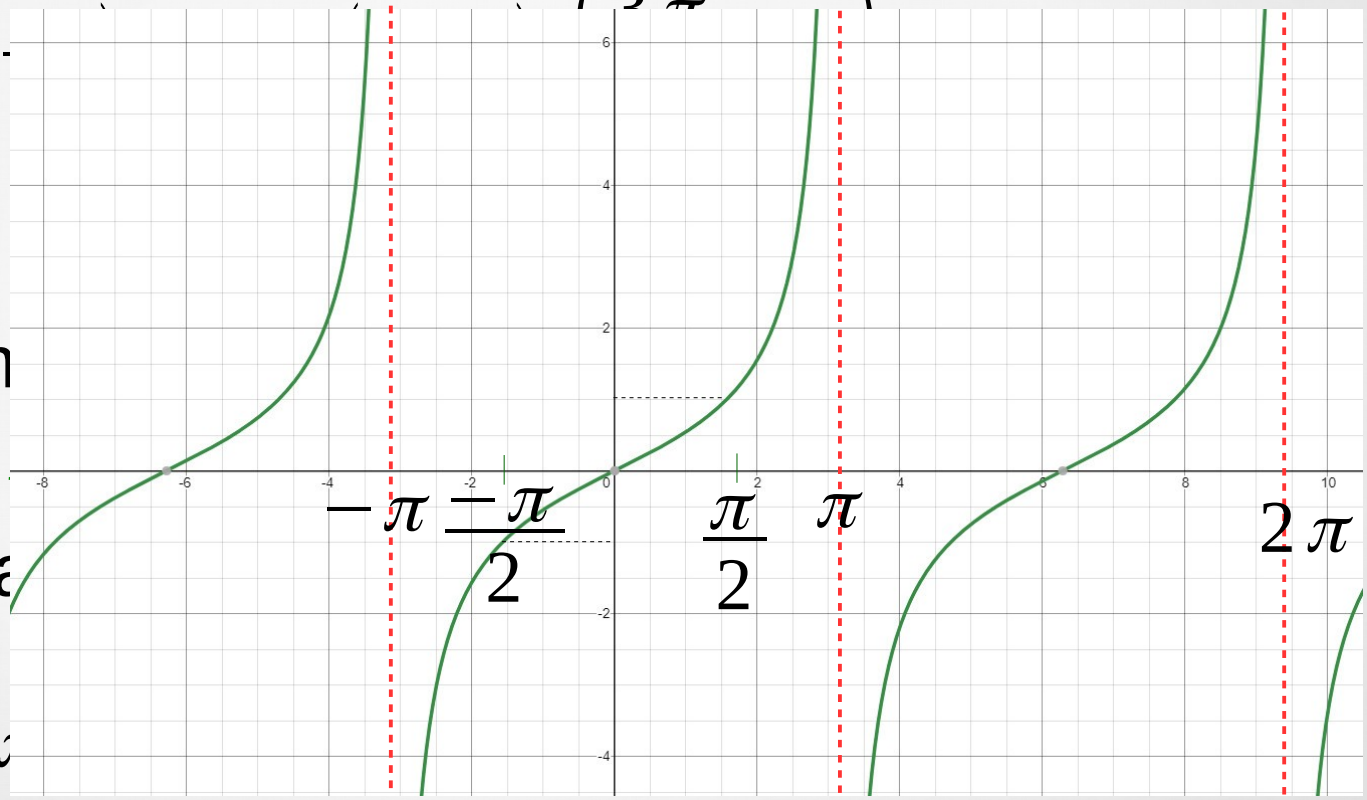
$\ldots,(-2\pi,-$

concave down:

$\ldots,(-\pi,0),(\pi,2\pi),(3\pi,4\pi),\ldots$