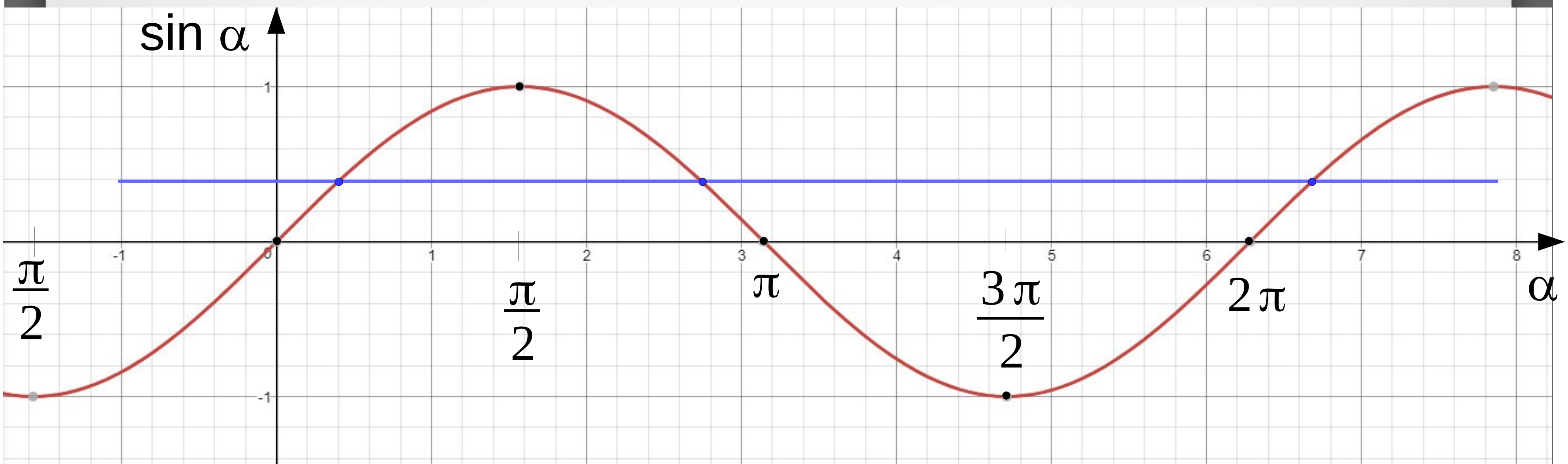
Inverse Trigonometric Functions

Learning objectives: in this section, we will:

- Understand and use the inverse sine, cosine, and tangent functions.
- Find the exact value of expressions involving the inverse sine, cosine, and tangent functions.
- Use a calculator to evaluate inverse trigonometric functions.
- Find exact values of composite functions with inverse trigonometric functions.

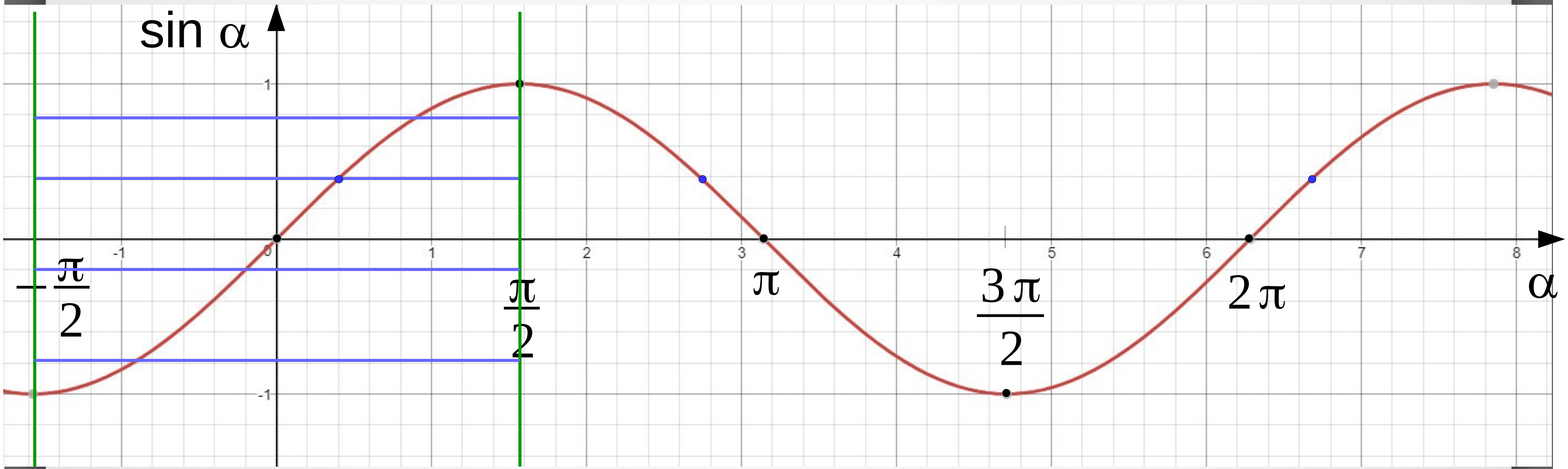
Inverse Trigonometric Functions

Sine function does not have inverse, because it fails *horizontal line test*.



Inverse Trigonometric Functions

Sine function does not have inverse, because it fails *horizontal line test*.

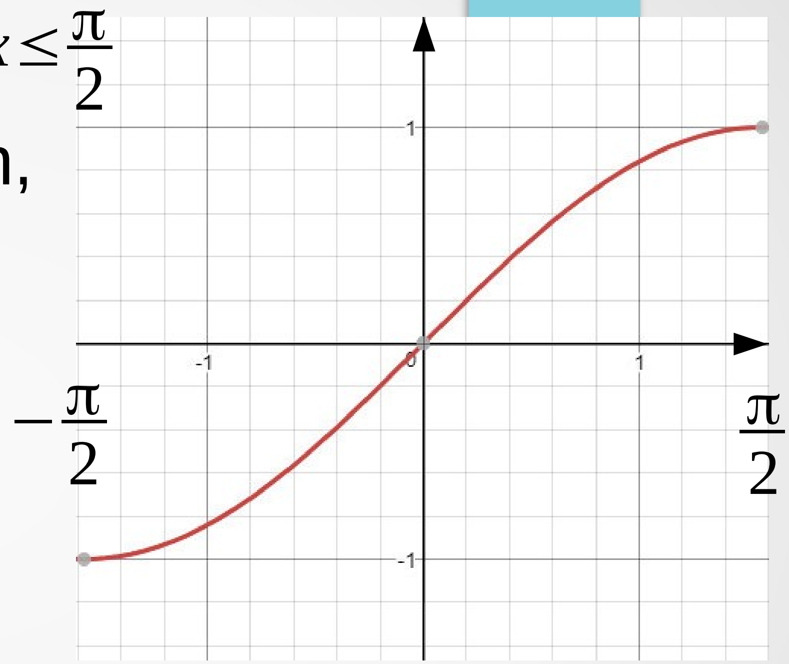


However, if we restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then every horizontal line drawn between -1 and 1 intersects the graph exactly once.

Inverse Trigonometric Functions

On the restricted domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$y = sin\ x$ has an inverse function, called *inverse sine function*.



notations:

$y = \sin^{-1} x$ $\qquad$ $y = \arcsin x$

note the difference:

$$y = \sin^{-1} x$$

inverse sine function

$$y = (\sin x)^{-1} = \frac{1}{\sin x} = csc\ x$$

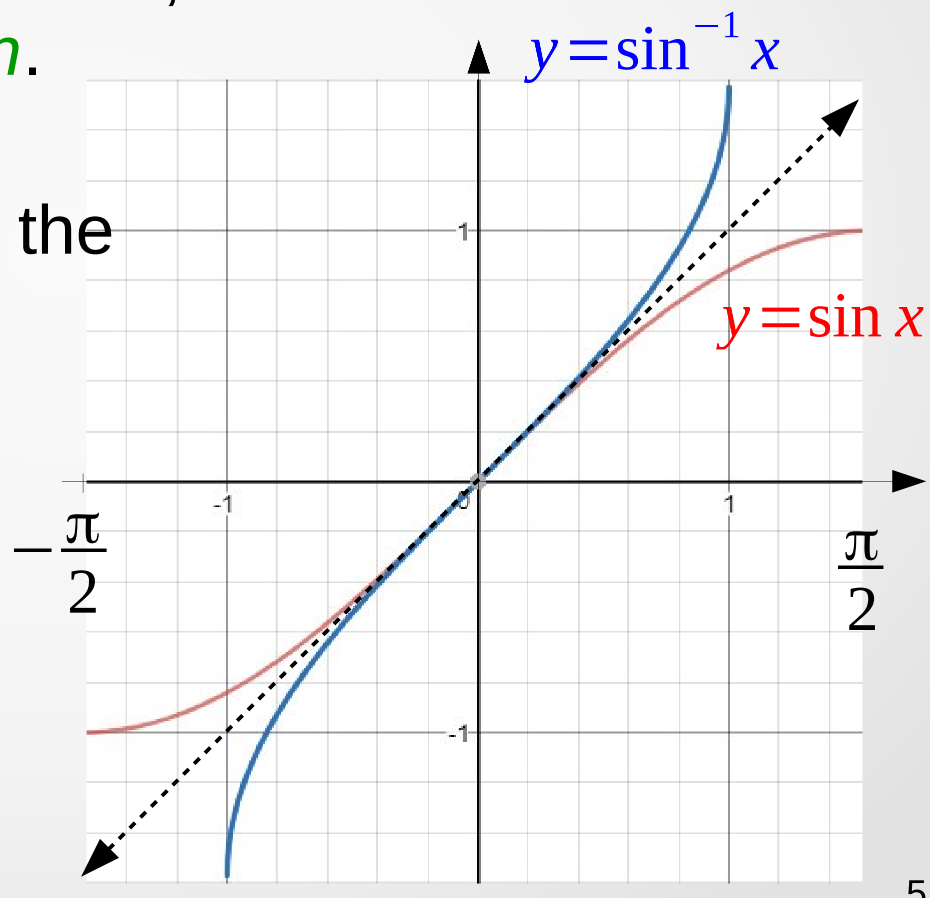
reciprocal of sine function

Inverse Trigonometric Functions

On the restricted domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$y = sin\ x$ has an inverse function, called *inverse sine function*.

graphs are reflected about the $y = x$ line



	$y = \sin^{-1} x$	$y = \sin x$
domain:	[-1,1]	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
range:	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	[-1,1]

Inverse Trigonometric Functions

Example: find $\sin^{-1}\left(-\frac{1}{2}\right)$

α	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\alpha$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Inverse Trigonometric Functions

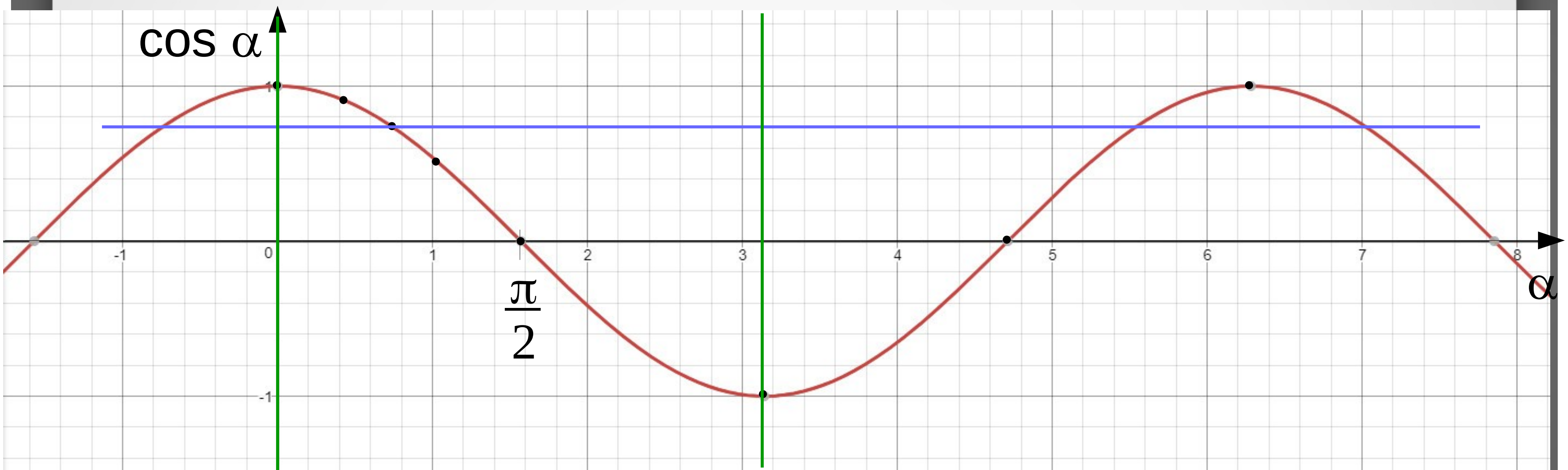
Example: find $\sin^{-1}\left(-\frac{1}{2}\right)$

Solution: $\sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$ because $\sin\left(-\frac{\pi}{6}\right)=-\frac{1}{2}$

α	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin α	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Inverse Trigonometric Functions

Similar to sine function, cosine function does not have inverse, because it fails *horizontal line test*.



However, if we restrict the domain to $[0,\pi]$, then every horizontal line drawn between -1 and 1 intersects the graph exactly once.

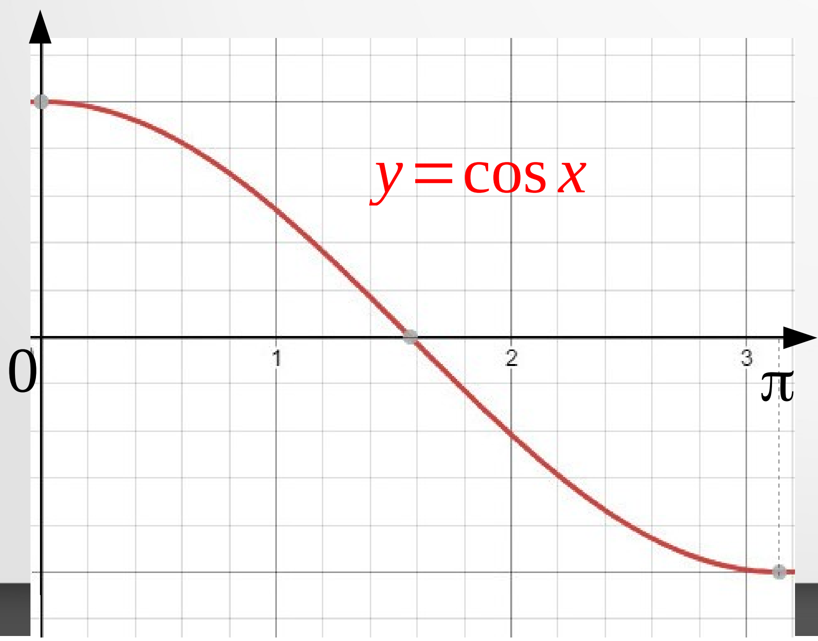
Inverse Trigonometric Functions

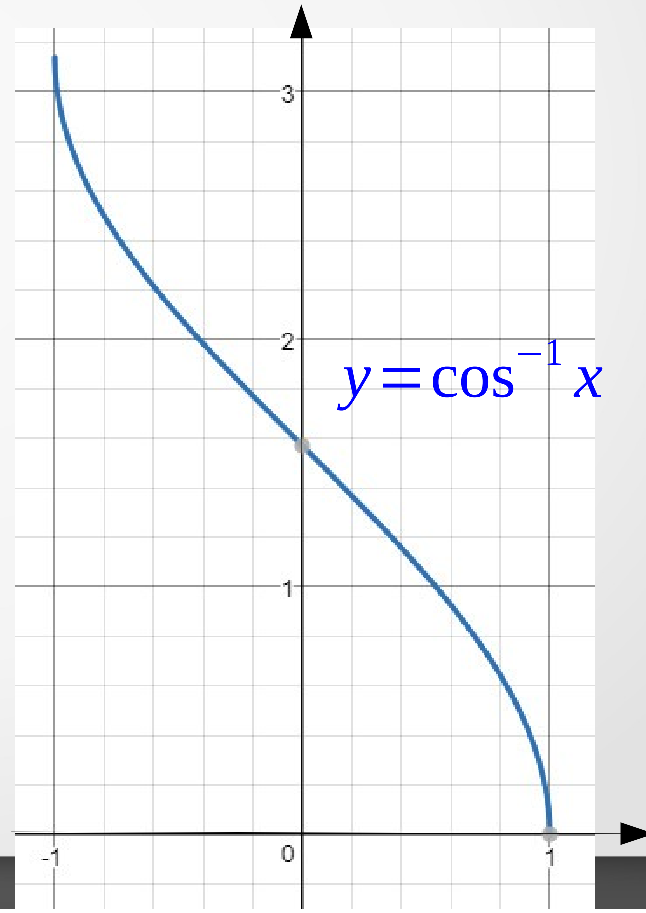
On the restricted domain $0 \leq x \leq \pi$

$y = cos\, x$ has an inverse function,
called *inverse cosine function*.

notations:

$y = \cos^{-1} x$ $\quad\quad$ $y = \arccos x$





Inverse Trigonometric Functions

Example: find $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
cos α	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Inverse Trigonometric Functions

Example: find $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

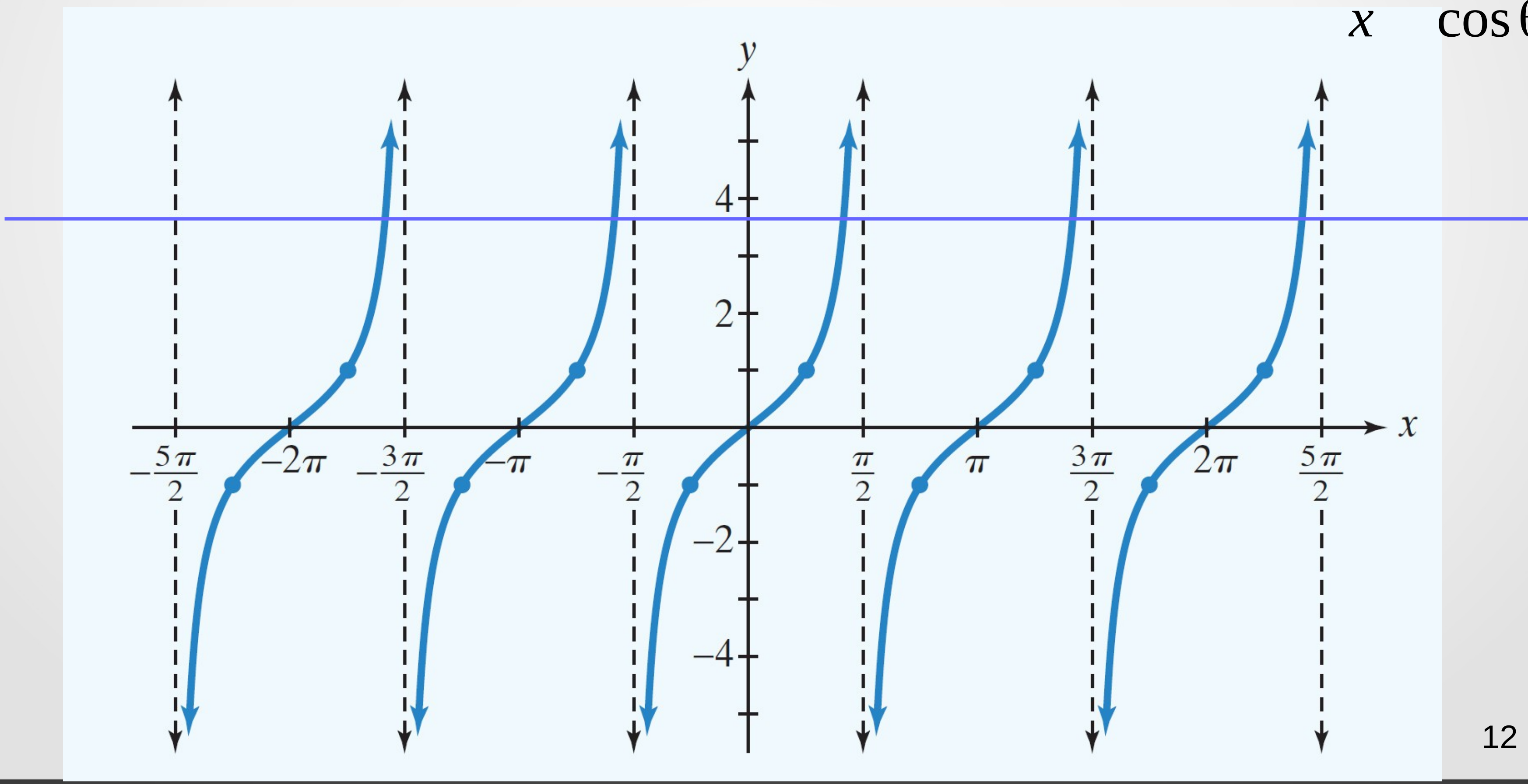
Solution: $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$, because $\cos^{-1}\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
cos α	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Inverse Trigonometric Functions

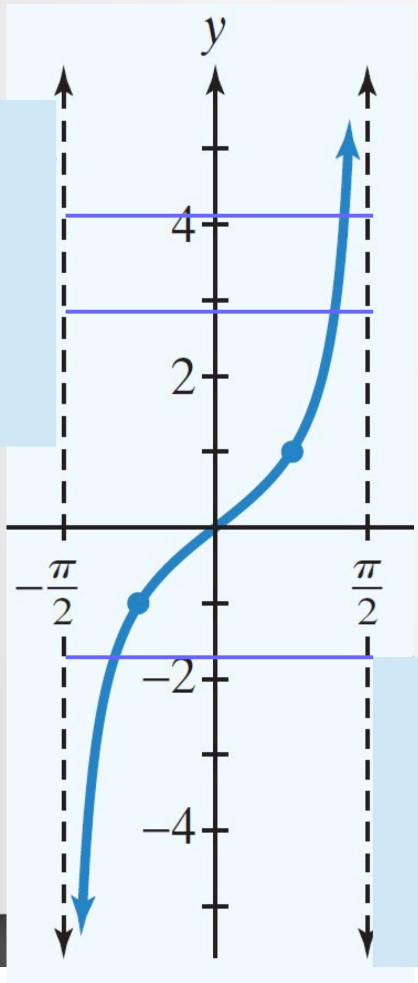
Tangent function does not have inverse either, because it fails *horizontal line test*.

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$



Inverse Trigonometric Functions

However, if we restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then every horizontal line intersects the graph exactly once.

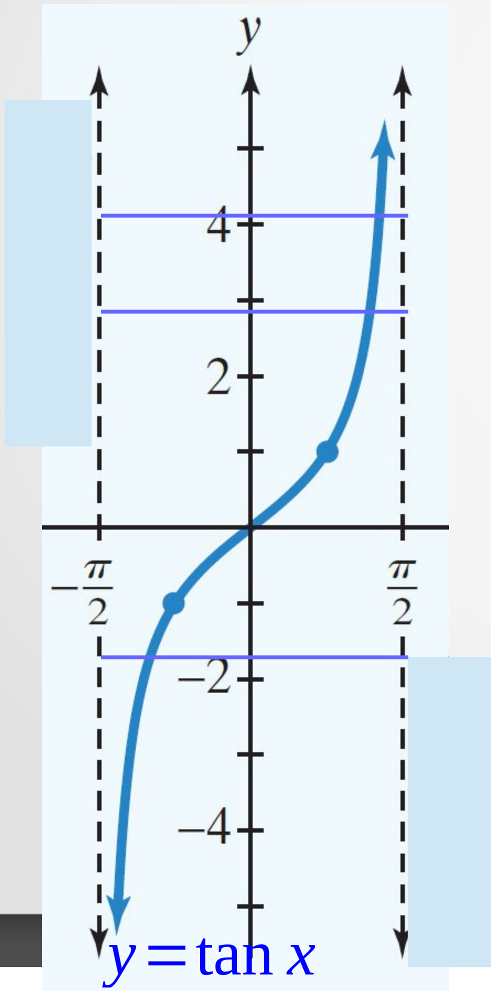


θ	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
tan θ	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

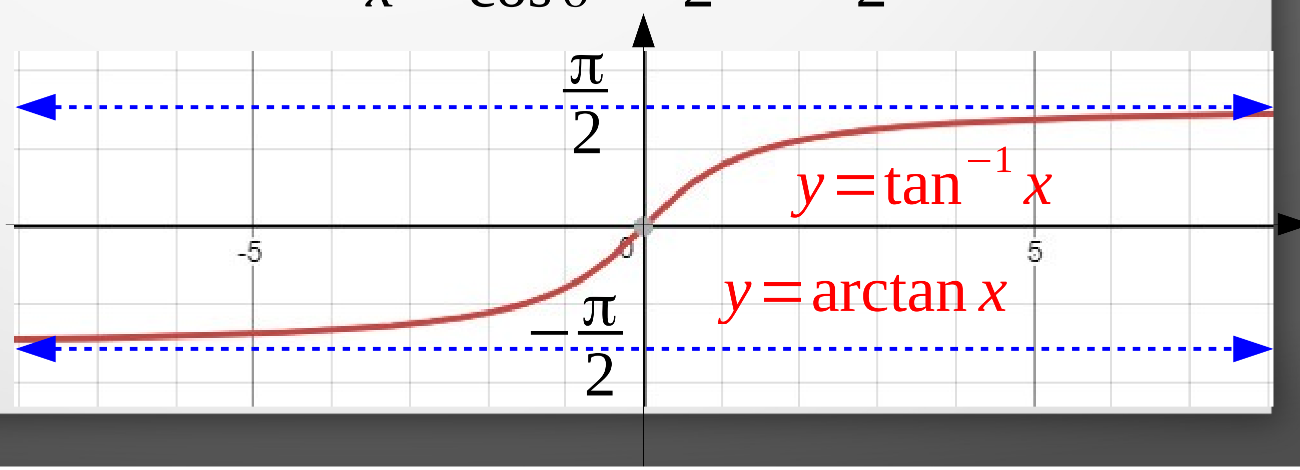
Inverse Trigonometric Functions

However, if we restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then every horizontal line intersects the graph exactly once.



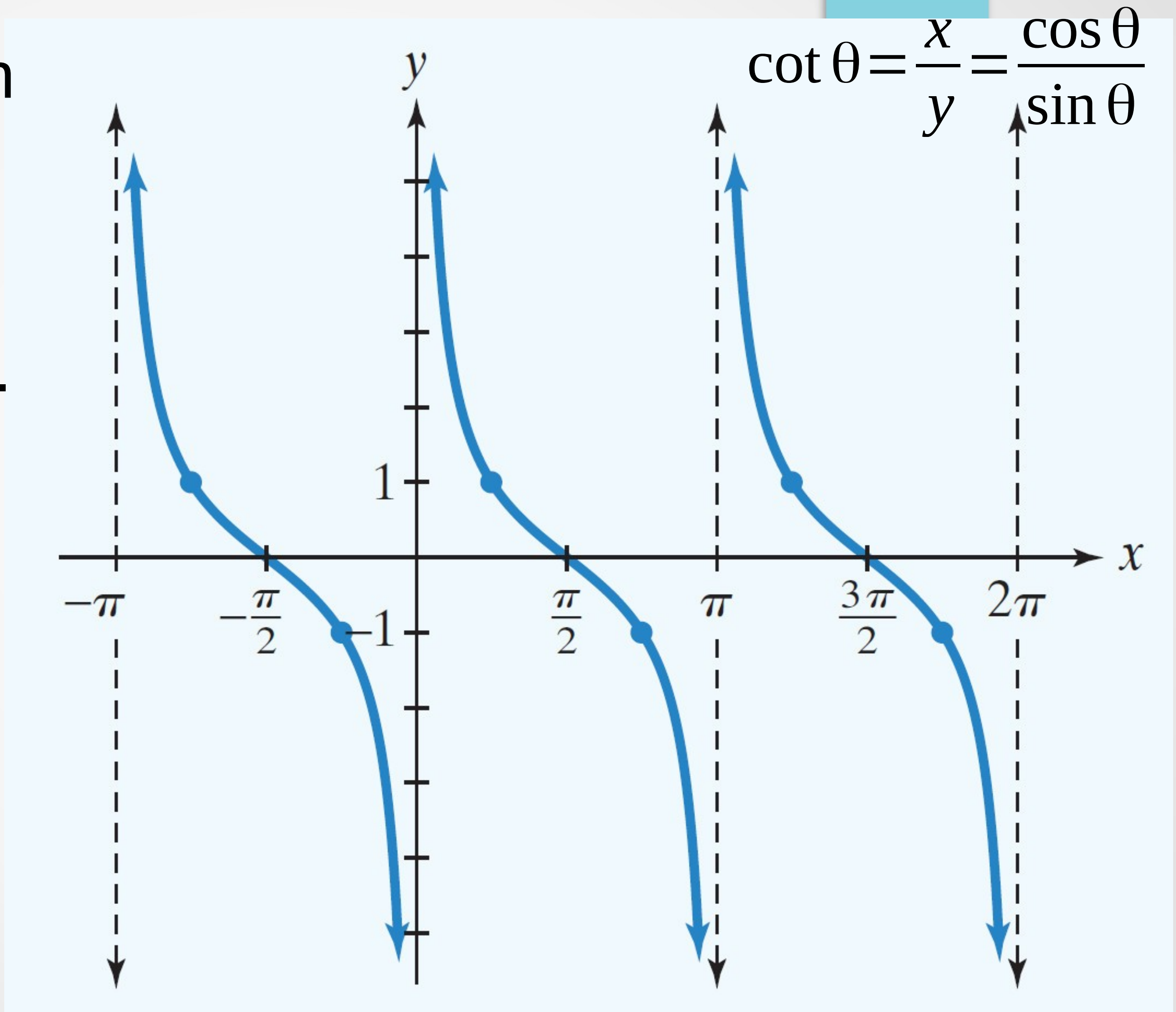
θ	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
tan θ	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$



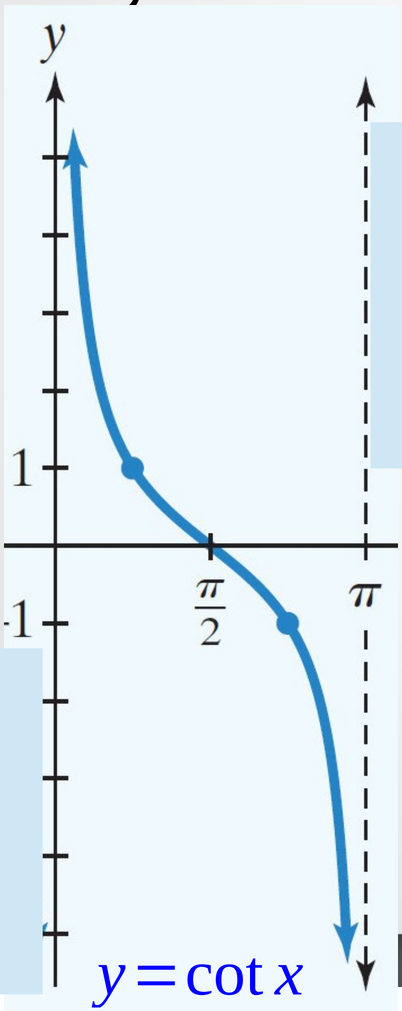
Inverse Trigonometric Functions

Cotangent function does not have inverse either, because it fails *horizontal line test*.



Inverse Trigonometric Functions

However, if we restrict the domain to $[0, \pi]$, then every horizontal line intersects the graph exactly once.

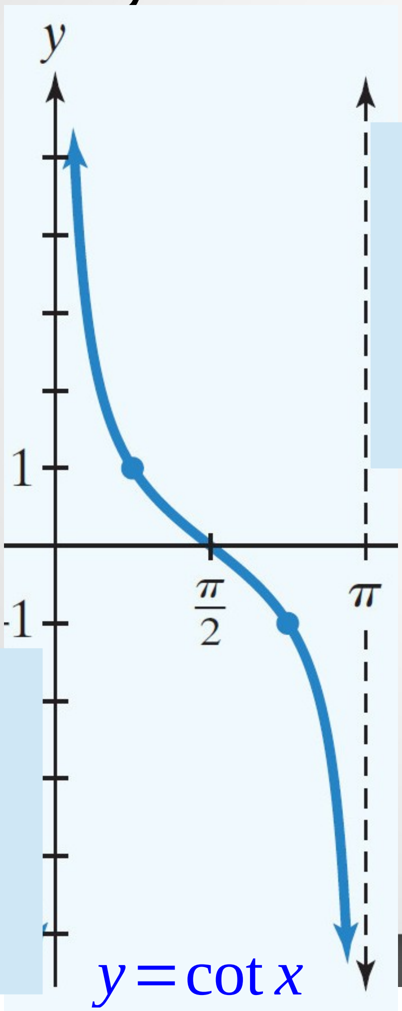


θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
$\cot\theta$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$

$$\cot\theta = \frac{x}{y} = \frac{\cos\theta}{\sin\theta}, 0 < \theta < \pi$$

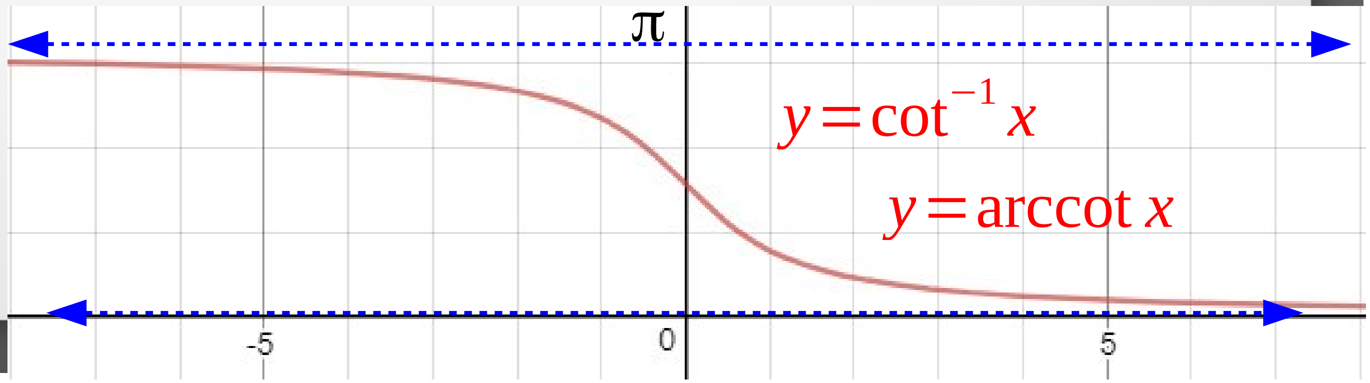
Inverse Trigonometric Functions

However, if we restrict the domain to $[0, \pi]$, then every horizontal line intersects the graph exactly once.



θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
$\cot\theta$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$

$$\cot\theta = \frac{x}{y} = \frac{\cos\theta}{\sin\theta}, 0 < \theta < \pi$$



Inverse Trigonometric Functions

Inverse properties

$\sin(\sin^{-1} x) = x$ for every x in the interval $[-1,1]$

$\sin^{-1}(\sin x) = x$ for every x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\cos(\cos^{-1} x) = x$ for every x in the interval $[-1,1]$

$\cos^{-1}(\cos x) = x$ for every x in the interval $[0, \pi]$

$\tan(\tan^{-1} x) = x$ for every real number x

$\tan^{-1}(\tan x) = x$ for every x in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Inverse Trigonometric Functions

Examples: find

1) $\tan^{-1} 1$

2) $\sin^{-1}\left(\sin \frac{\pi}{3}\right)$

3) $\cos\left(\sin^{-1} \frac{1}{2}\right)$

4) $sec\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

α	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin α	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
cos α	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

θ	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
tan θ	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Inverse Trigonometric Functions

Examples: find

1) $\tan^{-1} 1 = \frac{\pi}{4}$

2) $\sin^{-1}(\sin\frac{\pi}{3}) = \frac{\pi}{3}$

3) $\cos(\sin^{-1}\frac{1}{2})$

$= \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

4) $sec(\sin^{-1}\left(-\frac{1}{2}\right))$

$= sec\left(-\frac{\pi}{6}\right) = sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{6}\right)} = \frac{2\sqrt{3}}{3}$

even

α	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\alpha$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

θ	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\tan\theta$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

In-class practice

Exercises: find

1) $\sin^{-1} 0$

2) $\sin\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)$

3) $\tan^{-1}\left(\tan\left(-\frac{\pi}{3}\right)\right)$

4) $\cos^{-1}(\cos 2\pi)$

α	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin α	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

θ	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
tan θ	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

In-class practice

Exercises: find

1) $\sin^{-1} 0 = \frac{\pi}{4}$

2) $\sin\left(\sin^{-1}\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$

3) $\tan^{-1}\left(\tan\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$

out of interval

4) $\cos^{-1}(\cos 2\pi)$

$= \cos^{-1}(1) = 0$

α	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin α	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

θ	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
tan θ	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

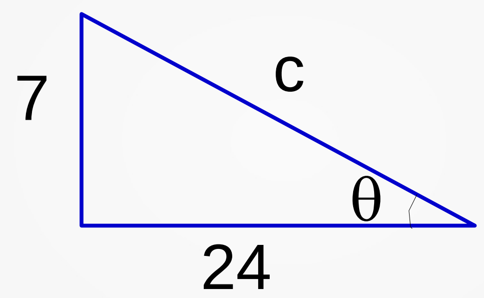
Inverse Trigonometric Functions

Example: find $\sin\left(\tan^{-1}\frac{7}{24}\right)$

Inverse Trigonometric Functions

Example: find $\sin\left(\tan^{-1}\frac{7}{24}\right)$

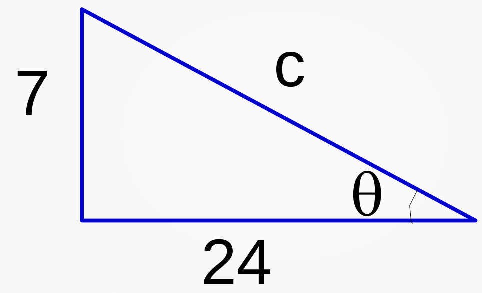
Using right triangle trigonometry:



Inverse Trigonometric Functions

Example: find $\sin\left(\tan^{-1}\frac{7}{24}\right)$

Using right triangle trigonometry:

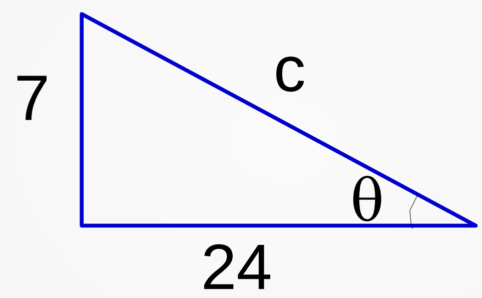


1st step: find hypotenuse c: $c = \sqrt{24^2 + 7^2} = \sqrt{625} = 25$

Inverse Trigonometric Functions

Example: find $\sin\left(\tan^{-1}\frac{7}{24}\right)$

Using right triangle trigonometry:



1st step: find hypotenuse c: $c=\sqrt{24^2+7^2}=\sqrt{625}=25$

2nd step: use definition of the sine function:

$$\sin\theta=\frac{7}{c}=\frac{7}{25}$$

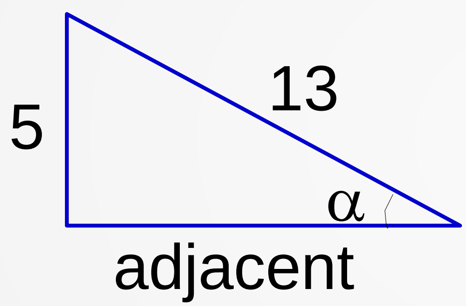
In-class practice

Exercise: find $\cot\left(\sin^{-1}\frac{5}{13}\right)$

In-class practice

Exercise: find $\cot\left(\sin^{-1}\frac{5}{13}\right)$

Using right triangle trigonometry:



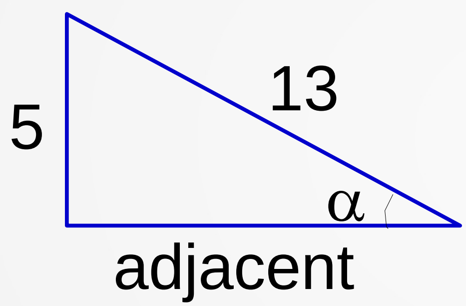
1st step: find missing side:

2nd step: use definition of the cotangent function:

In-class practice

Exercise: find $\cot\left(\sin^{-1}\frac{5}{13}\right)$

Using right triangle trigonometry:



1st step: find missing side: adjacent = $\sqrt{13^2-5^2}=\sqrt{144}=12$

2nd step: use definition of the cotangent function:

$$\cot(\alpha)=\frac{adjacent}{5}=\frac{12}{5}$$

Inverse Trigonometric Functions

Learning objectives: in this section, we:

- Learned to use the inverse sine, cosine, and tangent functions.
- Found the exact value of expressions involving the inverse sine, cosine, and tangent functions.
- Used a calculator to evaluate inverse trigonometric functions.
- Found exact values of composite functions with inverse trigonometric functions.