Zeros of polynomial functions

In this section we will:

- Evaluate a polynomial using the Remainder Theorem.
- Use the Factor Theorem to solve a polynomial equation.
- Use the Rational Zero Theorem to find rational zeros.
- Find zeros of a polynomial function.
- Use the Linear Factorization Theorem to find polynomials with given zeros.
- Use Descartes' Rule of Signs.

Zeros of polynomial functions

Remainder Theorem

If the polynomial $f(x)$ is divided by $x\text{-}C$, then the remainder is $f(C)$.

Zeros of polynomial functions

Remainder Theorem

If the polynomial $f(x)$ is divided by x-C, then the remainder is $f(C)$.

Example from previous slides:

$$(6x^2 - 10x + 21) \div (x + 3) = (6x - 28)\,R\,105$$

Zeros of polynomial functions

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$$f(x)=6x^2-10x+21$$

Zeros of polynomial functions

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$$f(C)=f(-3)$$

Zeros of polynomial functions

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$$f(x)=6x^2-10x+21$$

$$f(C)=f(-3)=6\cdot(-3)^2-10\cdot(-3)+21=54+30+21=105$$

Zeros of polynomial functions

Remainder Theorem

If the polynomial $f(x)$ is divided by x-C, then the remainder is $f(C)$.

An exercise from previous slides:

$$(x^3-4x^2+x+6)\div(x+1)=x^2-5x+6$$

$$f(x)=x^3-4x^2+x+6$$

$$f(C)=f(-1)=(-1)^3-4(-1)^2+(-1)+6=-1-4-1+6=0$$

In-class practice

Using the *remainder theorem*, if possible, answer the following questions:

(1) find the remainder of the division

$$\left(x^3-4x^2+5x+3\right)\div\left(x-3\right)$$

(2) find the remainder of the division

$$\left(x^2+10x+21\right)\div\left(x+7\right)$$

(3) find the remainder of the division

$$\left(18x^4+9x^3+3x^2\right)\div\left(3x^2+1\right)$$

Zeros of polynomial functions

Factor Theorem

Let $f(x)$ be a polynomial

a) if $f(C)=0$ then x-C is a factor of $f(x)$

b) if x-C is a factor of $f(x)$ then $f(C)=0$

In-class practice

Exercise 1: Solve the equation $2x^3-3x^2-11x+6=0$ given that -2 is a zero of $f(x)=2x^3-3x^2-11x+6$.

Zeros of Polynomial Functions

The Rational Zero Theorem

If $f(x)=a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ has integer coefficients and p/q (reduced to lowest terms) is a rational zero of f, then p is a factor of the *constant term* a_0, and q is a factor of *leading coefficient* a_n .

We can use this theorem to find _possible rational zeros_ of

$$f(x) : \frac{\text{factors of } a_0}{\text{factors of } a_n} \begin{array}{l} \longleftarrow \textit{constant term} \\ \longleftarrow \textit{leading coefficient} \end{array}$$

Zeros of Polynomial Functions: The Rational Zero Theorem

Example: let's list all the *possible rational zeros* of the polynomial function $f(x)=2x^3-3x^2-11x+6$:

Zeros of Polynomial Functions: The Rational Zero Theorem

Example: let's list all the *possible rational zeros* of the polynomial function $f(x) = 2x^3 - 3x^2 - 11x + 6$:

constant term

$$\frac{\text{factors of } a_0}{\text{factors of } a_n} =$$

leading coefficient

Zeros of Polynomial Functions: The Rational Zero Theorem

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constant term

$$\frac{\text{factors of } a_0}{\text{factors of } a_n} = \frac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1, \pm 2} =$$

leading coefficient

Zeros of Polynomial Functions: The Rational Zero Theorem

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leading coefficient

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How can we test them?

Zeros of Polynomial Functions: The Rational Zero Theorem

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leading coefficient

How can we test them? evaluation or synthetic division

Zeros of Polynomial Functions: The Rational Zero Theorem

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Let's evaluate f(x) at above values:

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$$f(1)=2\cdot 1^3-3\cdot 1^2-11\cdot 1+6=2-3-11+6\neq 0$$

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$$f\left(\frac{1}{2}\right)=2\cdot\left(\frac{1}{2}\right)^3-3\cdot\left(\frac{1}{2}\right)^2-11\cdot\left(\frac{1}{2}\right)+6=\frac{2}{8}-\frac{3}{4}-\frac{11}{2}+6=0$$

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$\frac{1}{2}$ *is a zero of* $f(x)$

Zeros of Polynomial Functions: Factor Theorem

Example: $\frac{1}{2}\, is\, a\, zero\, of\; f(x) = 2x^3 - 3x^2 - 11x + 6$

Zeros of Polynomial Functions: Factor Theorem

Example: $\frac{1}{2}\,is\,a\,zero\,of\,f(x)=2x^3-3x^2-11x+6$

Then, let’s use synthetic division: $(2x^3-3x^2-11x+6)\div\left(x-\frac{1}{2}\right)$

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$\frac{1}{2}$	x^3	x^2	x	$const$
	2	-3	-11	6
	↓			
	2			
	x^2	x	$const$	r

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	↓	1	-1	-6
	2	-2	-12	0
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Zeros of Polynomial Functions: Factor Theorem

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Therefore, $(2x^3-3x^2-11x+6)\div\left(x-\frac{1}{2}\right)=(2x^2-2x-12)$

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next step for us will be to find the zeros of

Therefore, $(2x^3-3x^2-11x+6)\div\left(x-\frac{1}{2}\right)=(2x^2-2x-12)$

Zeros of Polynomial Functions

Example: So $\left(2x^3-3x^2-11x+6\right)=\left(x-\frac{1}{2}\right)\left(2x^2-2x-12\right)$

Let's use factoring to find the zeros of $\left(2x^2-2x-12\right)$:

Zeros of Polynomial Functions

Example: So $(2x^3-3x^2-11x+6)=\left(x-\frac{1}{2}\right)(2x^2-2x-12)$

Let's use factoring to find the zeros of $(2x^2-2x-12)$:

$$(2x^2-2x-12)=2(x^2-x-6)$$

Zeros of Polynomial Functions

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$$\left(2x^2-2x-12\right)=2\left(x^2-x-6\right)$$

$$x^2-x-6=0$$

Zeros of Polynomial Functions

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$$x^2-x-6=0$$

$$(x-3)(x+2)=0$$

Zeros of Polynomial Functions

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$$x^2-x-6=0$$

$$\left(x-3\right)\left(x+2\right)=0$$

$x-3=0$ or $x+2=0$

$x=3$ or $x=-2$

Zeros of Polynomial Functions

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We found all the zeros of the polynomial function

$$f(x)=2x^3-3x^2-11x+6 \;:\; -2,\frac{1}{2},3$$

In-class practice

Example: Find all real zeros of the polynomial function

$$f(x) = 2x^4 - 3x^3 - 15x^2 + 32x - 12$$

In-class practice

Example: Find all real zeros of the polynomial function

$$f(x)=2x^4-3x^3-15x^2+32x-12$$

Hint: try $x=2$

Zeros of Polynomial Functions

Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n \geq 1$ then the equation $f(x)=0$ has *at least one root* (complex or real).

Properties of Roots of Polynomial Functions

(1) If a polynomial equation is of degree n, then the equation has n *roots* (counting multiple roots separately)

(2) If $a+bi$ is a root of a polynomial equation with real coefficients ($b \neq 0$), then $a-bi$ is also a root.

Zeros of Polynomial Functions

Descartes's rule of signs

Let $f(x)=a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial function with real coefficients.

1) The number of *positive real zeros* of f is either:

   (a) the same as the number of sign changes of $f(x)$

   or

   (b) less than the number of sign changes of $f(x)$ by a positive even integer.

   If f has exactly one variation in sign, then f has exactly one positive real zero.

2) The number of *negative real zeros* of f is either:

   (a) the same as the number of sign changes of $f(-x)$

   or

   (b) less than the number of sign changes of $f(-x)$ by a positive even integer.

   If $f(-x)$ has only one variation in sign, then f has exactly one negative real zero.

Zeros of Polynomial Functions

Example: Consider the polynomial function

$$f(x) = 4x^8 - 3x^5 + 7x^3 - 4x^4 - 3$$

Zeros of Polynomial Functions

Example: Consider the polynomial function

$$f(x)=4x^8-3x^5+7x^3-4x^4-3$$

1) there are 3 positive real zeros or 3-2 = 1 positive real zero.

Zeros of Polynomial Functions

Example: Consider the polynomial function

$$f(x) = 4x^8 - 3x^5 + 7x^3 - 4x^4 - 3$$

1) there are 3 positive real zeros or 3-2 = 1 positive real zero.

$$f(-x) = 4(-x)^8 - 3(-x)^5 + 7(-x)^3 - 4(-x)^4 - 3 =$$

$$= 4x^8 + 3x^5 - 7x^3 - 4x^4 - 3$$

Zeros of Polynomial Functions

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$$=4x^8+3x^5-7x^3-4x^4-3$$

2) there is exactly one negative real zero

Zeros of Polynomial Functions

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$$= 4x^8 + 3x^5 - 7x^3 - 4x^4 - 3$$

2) there is exactly one negative real zero

In-class practice

Exercise: Consider the polynomial function

$$f(x) = -4x^8 + 7x^7 + 10x^2 + 5x$$

Use the Descartes's rule of signs to estimate the number of positive and negative real zeros of f.