Zeros of Polynomial Functions

Today we will discuss:

- The Rational Zero Theorem
- Fundamental Theorem of Algebra
- Properties of Roots of Polynomial Functions
- Linear Factorization Theorem
- Descarte’s Rule of Signs

Zeros of Polynomial Functions

The Rational Zero Theorem

If $f(x)=a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ has integer coefficients and p/q (reduced to lowest terms) is a rational zero of f, then p is a factor of the *constant term* a_0, and q is a factor of *leading coefficient* a_n.

We can use this theorem to find *possible rational zeros* of

$$f(x) \; : \; \frac{\text{factors of } a_0}{\text{factors of } a_n} \begin{array}{l} \longleftarrow \textit{constant term} \\ \longleftarrow \textit{leading coefficient} \end{array}$$

Zeros of Polynomial Functions

The Rational Zero Theorem

Example: let’s list all the *possible rational zeros* of the polynomial function $f(x)=2x^3-3x^2-11x+6$:

Zeros of Polynomial Functions

The Rational Zero Theorem

Example: let's list all the *possible rational zeros* of the polynomial function $f(x) = \enclose{circle}{2}x^3 - 3x^2 - 11x \enclose{circle}{+6}$:

constant term

$$\frac{\text{factors of } a_0}{\text{factors of } a_n} =$$

leading coefficient

Zeros of Polynomial Functions

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Example: let's list all the *possible rational zeros* of the polynomial function $f(x)=2x^3-3x^2-11x+6$:

constant term

$$\frac{\text{factors of } a_0}{\text{factors of } a_n} = \frac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1, \pm 2} =$$

leading coefficient

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The Rational Zero Theorem

Example: let's list all the *possible rational zeros* of the polynomial function $f(x)=2x^3-3x^2-11x+6$:

constant term

$$\frac{\text{factors of } a_0}{\text{factors of } a_n} = \frac{\pm 1,\pm 2,\pm 3,\pm 6}{\pm 1,\pm 2}=\pm 1,\pm\frac{1}{2},\pm 2,\pm\frac{3}{2},\pm 3,\pm 6$$

leading coefficient

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leading coefficient

How can we test them?

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leading coefficient

How can we test them? evaluation or synthetic division

Zeros of Polynomial Functions

The Rational Zero Theorem

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$$\pm 1, \pm \frac{1}{2}, \pm 2, \pm \frac{3}{2}, \pm 3, \pm 6$$

Let’s evaluate f(x) at above values:

Zeros of Polynomial Functions

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Let’s evaluate f(x) at above values:

$$f(1) = 2 \cdot 1^3 - 3 \cdot 1^2 - 11 \cdot 1 + 6 = 2 - 3 - 11 + 6 \neq 0$$

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Let's evaluate f(x) at above values:

$$f(1)=2\cdot 1^3-3\cdot 1^2-11\cdot 1+6=2-3-11+6\neq 0$$

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$$f\left(\frac{1}{2}\right)=2\cdot\left(\frac{1}{2}\right)^3-3\cdot\left(\frac{1}{2}\right)^2-11\cdot\left(\frac{1}{2}\right)+6=\frac{2}{8}-\frac{3}{4}-\frac{11}{2}+6=0$$

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$\frac{1}{2}$ $is\,a\,zero\,of\,f(x)$

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Example: $\frac{1}{2}\ is\ a\ zero\ of\ f(x)=2x^3-3x^2-11x+6$

Then, let’s use synthetic division: $(2x^3-3x^2-11x+6)\div\left(x-\frac{1}{2}\right)$

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	x^3	x^2	x	$const$
$\frac{1}{2}$	2	-3	-11	6
	↓			
	2			
	x^2	x	$const$	r

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	x^3	x^2	x	$const$
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	$\downarrow$	1	-1	-6
	2	-2	-12	0
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Therefore, $(2x^3-3x^2-11x+6)\div\left(x-\frac{1}{2}\right)=(2x^2-2x-12)$

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next step for us will be to find the zeros of

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Zeros of Polynomial Functions

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Let’s use factoring to find the zeros of $(2x^2-2x-12)$:

Zeros of Polynomial Functions

The Rational Zero Theorem

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Let’s use factoring to find the zeros of $(2x^2-2x-12)$:

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$$x^2-x-6=0$$

Zeros of Polynomial Functions

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$$(x-3)(x+2)=0$$

Zeros of Polynomial Functions

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$x-3=0$ or $x+2=0$

$x=3$ or $x=-2$

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$x-3=0$ or $x+2=0$

$x=3$ or $x=-2$

We found all the zeros of the polynomial function

$$f(x)=2x^3-3x^2-11x+6\ :\ -2,\frac{1}{2},3$$

Zeros of Polynomial Functions

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Possible rational zeros: $\frac{\pm1,\pm2}{\pm1,\pm2}=$

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a_n a_0

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$$f(1)=2\cdot1^3+6\cdot1^2+5\cdot1+2\neq0$$

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$$\begin{array}{c|cccc}
 & x^3 & x^2 & x & const \\
-\frac{1}{2} & 2 & 6 & 5 & 2 \\
 & \downarrow & & & \\
\hline
 & 2 & & & \\
 & x^2 & x & const & r
\end{array}$$

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	x^3	x^2	x	$const$
$-\frac{1}{2}$	2	6	5	2
	↓	-1	-2.5	-1.25
	2	5	2.5	0.75
	x^2	x	$const$	r

Zeros of Polynomial Functions

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Let’s use synthetic division to check if

2 is a zero of *f(x)* :

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	↓			
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Possible rational zeros: $\dfrac{\pm 1, \pm 2}{\pm 1, \pm 2} = \pm 1, \pm \dfrac{1}{2}, \pm 2$

Let’s use synthetic division to check if 2 is a zero of *f(x)* :

	x^3	x^2	x	$const$
2	2	6	5	2
	↓	4	20	50
	2	10	25	52
	x^2	x	$const$	r

Zeros of Polynomial Functions

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$$\underset{\overset{\uparrow}{a_n}}{2}\qquad\qquad \underset{\overset{\uparrow}{a_0}}{2}$$

Possible rational zeros: $\dfrac{\pm1,\pm2}{\pm1,\pm2}=\pm1,\pm\dfrac{1}{2},\pm2$

Let’s use synthetic division to check if -2 is a zero of *f(x)* :

$$\begin{array}{c|cccc} & x^3 & x^2 & x & const \\ -2 & 2 & 6 & 5 & 2 \\ & \downarrow & & & \\ \hline & 2 & & & \\ & x^2 & x & const & r \end{array}$$

Zeros of Polynomial Functions

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Possible rational zeros: $\dfrac{\pm 1,\pm 2}{\pm 1,\pm 2}=\pm 1,\pm\dfrac{1}{2},\pm 2$

Let's use synthetic division to check if

-2 is a zero of *f(x)* :

yes, it is!

	x^3	x^2	x	$const$
-2	2	6	5	2
	↓	-4	-4	-2
	2	2	1	0
	x^2	x	$const$	r

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-2 is a zero of *f(x)* :

$$2x^3+6x^2+5x+2=$$

$$=(x-(-2))(2x^2+2x+1)$$

$$\begin{array}{r|rrrr} -2 & 2 & 6 & 5 & 2 \\ & \downarrow & -4 & -4 & -2 \\ \hline & 2 & 2 & 1 & 0 \\ & x^2 & x & const & r \end{array}$$

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let’s solve: $2x^2+2x+1=0$

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let's solve: $2x^2+2x+1=0$

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=$$

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$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-2\pm\sqrt{2^2-4\cdot2\cdot1}}{2\cdot2}=\frac{-2\pm\sqrt{-4}}{4}=$$

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-2	2	6	5	2
	$\downarrow$	-4	-4	-2
	2	2	1	0
	x^2	x	$const$	r

let's solve: $2x^2+2x+1=0$

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-2\pm\sqrt{2^2-4\cdot2\cdot1}}{2\cdot2}=\frac{-2\pm\sqrt{-4}}{4}=\frac{-2\pm2i}{4}=\frac{-1\pm i}{2}$$

no more real number zeros, 2 complex number zeros

Zeros of Polynomial Functions

The Rational Zero Theorem

Example 2: let's find all the zeros of the polynomial function $f(x)=2x^3+6x^2+5x+2$

Answer: *f(x)* has one real number zero, -2, and two complex number zeros: $\frac{-1\pm i}{2}$.

Zeros of Polynomial Functions

Fundamental Theorem o Algebra

If $f(x)$ is a polynomial of degree n, where $n \geq 1$ then the equation $f(x)=0$ has *at least one root* (complex or real).

Properties of Roots of Polynomial Functions

(1) If a polynomial equation is of degree n, then the equation has *n roots* (counting multiple roots separately)

(2) If $a+bi$ is a root of a polynomial equation with real coefficients ($b \neq 0$), then $a\text{-}bi$ is also a root.

Zeros of Polynomial Functions

Linear Factorization Theorem

Let $f(x)=a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, $n \geq 1$ and $a_n \neq 0$ then $f(x)=a_n(x-c_1)(x-c_2)\ldots(x-c_n)$, where $c_1, c_2, \ldots, c_n$ are *roots* of *f(x)*, possibly *complex*, *real* and *repeating*.

Zeros of Polynomial Functions

Example: Find the n^{th} degree polynomial function with real coefficients satisfying the given conditions:

$n = 3$,
zeros: $-5,\ 4+3i$
$f(2) = 91$

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Solution: using the *Linear Factorization Theorem*,

$$f(x) = a_3(x - c_1)(x - c_2)(x - c_3)$$

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$$f(x) = a_3(x - c_1)(x - c_2)(x - c_3)$$

In addition, since we have $4+3i$ as a root, then $4-3i$ should also be a root.

Zeros of Polynomial Functions

Example: Find the n^{th} degree polynomial function with real coefficients satisfying the given conditions:

$n = 3$,
zeros: -5, $4+3i$
$f(2) = 91$

Solution: using the *Linear Factorization Theorem*,

$$f(x)=a_3(x-c_1)(x-c_2)(x-c_3)=a_3(x+5)(x-4-3i)(x-4+3i)$$

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$$=a_3(x^3-3x^2-15x+125)$$

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Example: Find the n^{th} degree polynomial function with real coefficients satisfying the given conditions:

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$$91 = a_3 \cdot 91$$

$$1 = a_3$$

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Example: Find the n^{th} degree polynomial function with real coefficients satisfying the given conditions:

$n = 3$,
zeros: $-5,\ 4+3i$
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Solution: ... $f(x)=a_3(x^3-3x^2-15x+125)$

Next, let's use $f(2) = 91$: $91=a_3(2^3-3\cdot2^2-15\cdot2+125)$

$$91=a_3\cdot91$$

$$1=a_3$$

Therefore, $f(x)=1(x^3-3x^2-15x+125)=x^3-3x^2-15x+125$

In-class practice

Exercise: Find the n^{th} degree polynomial function with real coefficients satisfying the given conditions:

$n = 4$,
zeros: $-4,\ 2+3i,\ \frac{1}{2}$
$f(-1) = 216$

In-class practice

Exercise: Find the n^{th} degree polynomial function with real coefficients satisfying the given conditions:

$n = 4$,
zeros: $-4,\ 2+3i,\ \frac{1}{2}$
$f(-1) = 216$

Answer:

$$f(x) = -3\left(x^4 - \frac{1}{3}x^3 - 3x^2 + 53x - \frac{52}{3}\right) = -3x^4 + x^3 + 3x^2 - 159x + 52$$

Zeros of Polynomial Functions

Descartes's rule of signs

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial function with real coefficients.

1) The number of *positive real zeros* of f is either:

   (a) the same as the number of sign changes of $f(x)$

   or

   (b) less than the number of sign changes of $f(x)$ by a positive even integer.

   If f has exactly one variation in sign, then f has exactly one positive real zero.

2) The number of *negative real zeros* of f is either:

   (a) the same as the number of sign changes of $f(-x)$

   or

   (b) less than the number of sign changes of $f(-x)$ by a positive even integer.

   If $f(-x)$ has only one variation in sign, then f has exactly one negative real zero.

Zeros of Polynomial Functions

Example: Consider the polynomial function

$$f(x)=4x^8-3x^5+7x^3-4x^4-3$$

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Zeros of Polynomial Functions

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$$f(x)=4x^8-3x^5+7x^3-4x^4-3$$

1) there are 3 positive real zeros or 3-2 = 1 positive real zero.

$$f(-x)=4(-x)^8-3(-x)^5+7(-x)^3-4(-x)^4-3=$$

$$=4x^8+3x^5-7x^3-4x^4-3$$

Zeros of Polynomial Functions

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2) there is exactly one negative real zero

In-class practice

Exercise: Consider the polynomial function

$$f(x) = -4x^8 + 7x^7 + 10x^2 + 5x$$

Use the Descartes's rule of signs to estimate the number of positive and negative real zeros of f.

Homework assignment

1) zyBooks: *review* Section 3.6 *Zeros of polynomial functions*

or

Textbook: *review* Section 2.5 *Zeros of Polynomial Functions*.

2) We will have Quiz 8 based on today's topics in the beginning of our next meeting.

3) WeBWorK: HW 8 (due date is in one week)