

Zeros of Polynomial Functions

Today we will discuss:

- The Rational Zero Theorem
- Fundamental Theorem of Algebra
- Properties of Roots of Polynomial Functions
- Linear Factorization Theorem
- Descartes's Rule of Signs

Zeros of Polynomial Functions

The Rational Zero Theorem

If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ has integer coefficients and p/q (reduced to lowest terms) is a rational zero of f , then p is a factor of the **constant term** a_0 , and q is a factor of **leading coefficient** a_n .

We can use this theorem to find possible rational zeros of

$f(x) : \frac{\text{factors of } a_0}{\text{factors of } a_n}$ ← **constant term**
← **leading coefficient**

Zeros of Polynomial Functions

The Rational Zero Theorem

Example: let's list all the possible rational zeros of the polynomial function $f(x) = 2x^3 - 3x^2 - 11x + 6$:

Zeros of Polynomial Functions

The Rational Zero Theorem

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constant term
↓
factors of a_0
=
factors of a_n
↑
leading coefficient

Zeros of Polynomial Functions

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constant term

factors of a_0 = $\frac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1, \pm 2}$ =

factors of a_n

leading coefficient

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leading coefficient

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leading coefficient

How can we test them?

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leading coefficient

How can we test them? evaluation or synthetic division

Zeros of Polynomial Functions

The Rational Zero Theorem

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Let's evaluate $f(x)$ at above values:

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Let's evaluate $f(x)$ at above values:

$$f(1) = 2 \cdot 1^3 - 3 \cdot 1^2 - 11 \cdot 1 + 6 = 2 - 3 - 11 + 6 \neq 0$$

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$$f\left(\frac{1}{2}\right) = 2 \cdot \left(\frac{1}{2}\right)^3 - 3 \cdot \left(\frac{1}{2}\right)^2 - 11 \cdot \left(\frac{1}{2}\right) + 6 = \frac{2}{8} - \frac{3}{4} - \frac{11}{2} + 6 = 0$$

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$\frac{1}{2}$ is a zero of $f(x)$

Zeros of Polynomial Functions

The Rational Zero Theorem

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Example: $\frac{1}{2}$ is a zero of $f(x) = 2x^3 - 3x^2 - 11x + 6$

Then, let's use synthetic division: $(2x^3 - 3x^2 - 11x + 6) \div \left(x - \frac{1}{2}\right)$

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$\frac{1}{2}$	x^3	x^2	x	<i>const</i>
	2	-3	-11	6
	↓			
	2			
	x^2	x	<i>const</i>	r

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$\frac{1}{2}$	x^3	x^2	x	<i>const</i>
	2	-3	-11	6
	↓	1	-1	-6
	2	-2	-12	0
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Therefore, $(2x^3 - 3x^2 - 11x + 6) \div \left(x - \frac{1}{2}\right) = (2x^2 - 2x - 12)$

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next step for us
will be to find the
zeros of

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Zeros of Polynomial Functions

The Rational Zero Theorem

Example:

Let's use factoring to find the zeros of $(2x^2 - 2x - 12)$:

Zeros of Polynomial Functions

The Rational Zero Theorem

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Let's use factoring to find the zeros of $(2x^2 - 2x - 12)$:

$$(2x^2 - 2x - 12) = 2(x^2 - x - 6)$$

Zeros of Polynomial Functions

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Let's use factoring to find the zeros of $(2x^2 - 2x - 12)$:

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$$x^2 - x - 6 = 0$$

Zeros of Polynomial Functions

The Rational Zero Theorem

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Let's use factoring to find the zeros of $(2x^2 - 2x - 12)$:

$$(2x^2 - 2x - 12) = 2(x^2 - x - 6)$$

$$x^2 - x - 6 = 0$$

$$(x - 3)(x + 2) = 0$$

Zeros of Polynomial Functions

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$$x^2 - x - 6 = 0$$

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$$x - 3 = 0 \quad \text{or} \quad x + 2 = 0$$

$$x = 3 \quad \text{or} \quad x = -2$$

Zeros of Polynomial Functions

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We found all the zeros of the polynomial function

$$f(x) = 2x^3 - 3x^2 - 11x + 6 : -2, \frac{1}{2}, 3$$

Zeros of Polynomial Functions

The Rational Zero Theorem

Example 2: let's find all the zeros of the polynomial function $f(x) = 2x^3 + 6x^2 + 5x + 2$

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Possible rational zeros: $\frac{\pm 1, \pm 2}{\pm 1, \pm 2} =$

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$$f(1) = 2 \cdot 1^3 + 6 \cdot 1^2 + 5 \cdot 1 + 2 \neq 0$$

Zeros of Polynomial Functions

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$$\begin{array}{c} \uparrow \qquad \qquad \qquad \uparrow \\ a_n \qquad \qquad \qquad a_0 \end{array}$$

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$$f\left(\frac{1}{2}\right) = 2 \cdot \left(\frac{1}{2}\right)^3 + 6 \cdot \left(\frac{1}{2}\right)^2 + 5 \cdot \left(\frac{1}{2}\right) + 2 = \frac{2}{8} + \frac{6}{4} + \frac{5}{2} + 2 \neq 0$$

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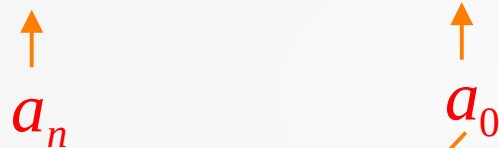
Let's use synthetic division to check if

$-\frac{1}{2}$ is a zero of $f(x)$:

Zeros of Polynomial Functions

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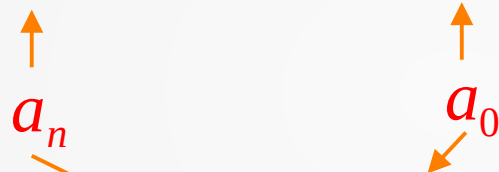
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	x^3	x^2	x	<i>const</i>
$-\frac{1}{2}$	2	6	5	2
	↓			
	2			
	x^2	x	<i>const</i>	<i>r</i>

Zeros of Polynomial Functions

The Rational Zero Theorem

Example 2: let's find all the zeros of the polynomial function $f(x) = 2x^3 + 6x^2 + 5x + 2$



Possible rational zeros: $\frac{\pm 1, \pm 2}{\pm 1, \pm 2} = \pm 1, \pm \frac{1}{2}, \pm 2$

Let's use synthetic division to check if $-\frac{1}{2}$ is a zero of $f(x)$:

$-\frac{1}{2}$	x^3	x^2	x	<i>const</i>
	2	6	5	2
↓		-1	-2.5	-1.25
	2	5	2.5	0.75
	x^2	x	<i>const</i>	<i>r</i>

Zeros of Polynomial Functions

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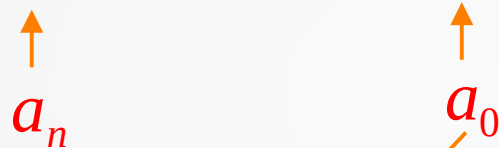
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2 is a zero of $f(x)$:

Zeros of Polynomial Functions

The Rational Zero Theorem

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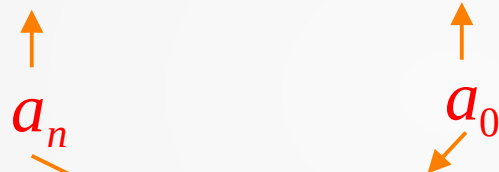
2 is a zero of $f(x)$:

	x^3	x^2	x	<i>const</i>
2	2	6	5	2
↓				
<hr/>				
	2			
	x^2	x	<i>const</i>	<i>r</i>

Zeros of Polynomial Functions

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Example 2: let's find all the zeros of the polynomial function $f(x) = 2x^3 + 6x^2 + 5x + 2$



Possible rational zeros: $\frac{\pm 1, \pm 2}{\pm 1, \pm 2} = \pm 1, \pm \frac{1}{2}, \pm 2$

Let's use synthetic division to check if

2 is a zero of $f(x)$:

	x^3	x^2	x	<i>const</i>
2	2	6	5	2
↓		4	20	50
	2	10	25	52
	x^2	x	<i>const</i>	<i>r</i>

Zeros of Polynomial Functions

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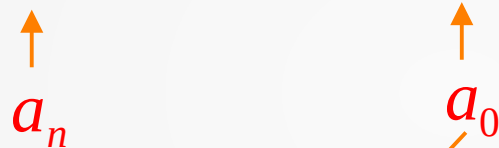
Let's use synthetic division to check if

-2 is a zero of $f(x)$:

Zeros of Polynomial Functions

The Rational Zero Theorem

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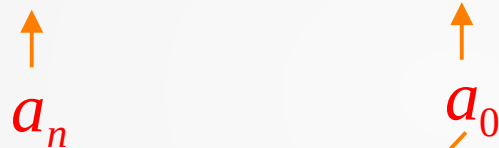
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Possible rational zeros: $\frac{\pm 1, \pm 2}{\pm 1, \pm 2} = \pm 1, \pm \frac{1}{2}, \pm 2$

Let's use synthetic division to check if

-2 is a zero of $f(x)$:

yes, it is!

	x^3	x^2	x	<i>const</i>
-2	2	6	5	2
↓		-4	-4	-2
	2	2	1	0
	x^2	x	<i>const</i>	<i>r</i>

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-2 is a zero of $f(x)$:

$$2x^3 + 6x^2 + 5x + 2 =$$

$$= (x - (-2))(2x^2 + 2x + 1)$$

-2	2	6	5	2
	↓	-4	-4	-2
	<hr/>			
	2	2	1	0
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let's solve: $2x^2 + 2x + 1 = 0$

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$$= (x - (-2))(2x^2 + 2x + 1)$$

let's solve: $2x^2 + 2x + 1 = 0$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} =$$

-2	2	6	5	2
	↓	-4	-4	-2
	2	2	1	0
	x^2	x	<i>const</i>	<i>r</i>

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let's solve: $2x^2 + 2x + 1 = 0$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} =$$

-2	2	6	5	2
	↓	-4	-4	-2
<hr/>				
	2	2	1	0
	x^2	x	const	r

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$$2x^3 + 6x^2 + 5x + 2 =$$

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let's solve: $2x^2 + 2x + 1 = 0$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \frac{-2 \pm \sqrt{-4}}{4} =$$

-2	2	6	5	2
	↓	-4	-4	-2
<hr/>				
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$$2x^3 + 6x^2 + 5x + 2 =$$

$$= (x - (-2))(2x^2 + 2x + 1)$$

let's solve: $2x^2 + 2x + 1 = 0$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \frac{-2 \pm \sqrt{-4}}{4} = \frac{-2 \pm 2i}{4} = \frac{-1 \pm i}{2}$$

no more real number zeros, 2 complex number zeros

-2	2	6	5	2
↓	-4	-4	-2	
<hr/>				
2	2	1	0	
x^2	x	const	r	

Zeros of Polynomial Functions

The Rational Zero Theorem

Example 2: let's find all the zeros of the polynomial function $f(x) = 2x^3 + 6x^2 + 5x + 2$

Answer: $f(x)$ has one real number zero, -2 , and two complex number zeros: $\frac{-1 \pm i}{2}$.

Zeros of Polynomial Functions

Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n , where $n \geq 1$ then the equation $f(x) = 0$ has at least one root (complex or real).

Properties of Roots of Polynomial Functions

- (1) If a polynomial equation is of degree n , then the equation has n roots (counting multiple roots separately)
- (2) If $a+bi$ is a root of a polynomial equation with real coefficients ($b \neq 0$), then $a-bi$ is also a root.

Zeros of Polynomial Functions

Linear Factorization Theorem

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, $n \geq 1$ and $a_n \neq 0$ then
 $f(x) = a_n (x - c_1)(x - c_2) \dots (x - c_n)$, where c_1, c_2, \dots, c_n are
roots of $f(x)$, possibly *complex, real and repeating*.

Zeros of Polynomial Functions

Example: Find the n^{th} degree polynomial function with real coefficients satisfying the given conditions:

$$n = 3,$$

$$\text{zeros: } -5, 4+3i$$

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$$f(x) = a_3(x - c_1)(x - c_2)(x - c_3)$$

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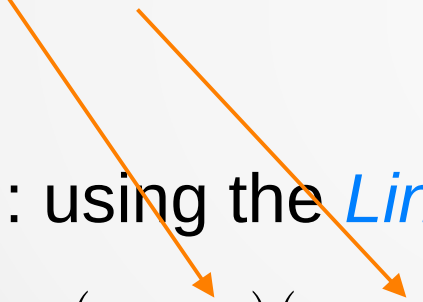
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$$f(x) = a_3(x - c_1)(x - c_2)(x - c_3) = a_3(x + 5)(x - 4 - 3i)(x - 4 + 3i)$$

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Next, let's use $f(2) = 91$: $91 = a_3(2^3 - 3 \cdot 2^2 - 15 \cdot 2 + 125)$

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Therefore, $f(x) = 1(x^3 - 3x^2 - 15x + 125) = x^3 - 3x^2 - 15x + 125$

In-class practice

Exercise: Find the n^{th} degree polynomial function with real coefficients satisfying the given conditions:

$$\begin{aligned} n &= 4, \\ \text{zeros: } &-4, 2+3i, \frac{1}{2} \\ f(-1) &= 216 \end{aligned}$$

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Answer:

$$f(x) = -3\left(x^4 - \frac{1}{3}x^3 - 3x^2 + 53x - \frac{52}{3}\right) = -3x^4 + x^3 + 3x^2 - 159x + 52$$

Zeros of Polynomial Functions

Descartes's rule of signs

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ be a polynomial function with real coefficients.

- 1) The number of positive real zeros of f is either:
 - (a) the same as the number of sign changes of $f(x)$
or
 - (b) less than the number of sign changes of $f(x)$ by a positive even integer.

If f has exactly one variation in sign, then f has exactly one positive real zero.

- 2) The number of negative real zeros of f is either:
 - (a) the same as the number of sign changes of $f(-x)$
or
 - (b) less than the number of sign changes of $f(-x)$ by a positive even integer.

If $f(-x)$ has only one variation in sign, then f has exactly one negative real zero.


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2) there is **exactly one negative real zero**

In-class practice

Exercise: Consider the polynomial function

$$f(x) = -4x^8 + 7x^7 + 10x^2 + 5x$$

Use the Descartes's rule of signs to estimate the number of positive and negative real zeros of f .

Homework assignment

1) zyBooks: *review Section 3.6 Zeros of polynomial functions*

or

Textbook: *review Section 2.5 Zeros of Polynomial Functions.*

2) We will have **Quiz 8** based on today's topics in the beginning of our next meeting.

3) WeBWorK: **HW 8** (due date is in one week)