Transformation of Functions

Transformation of Functions

Today we will work with graphs and functions and will discuss:

- horizontal and vertical shifts of graphs
- horizontal and vertical stretches of graphs
- horizontal and vertical shrinks/compressions of graphs
- reflections of graphs about the x-axis and y-axis

[Def] A *function transformation* is an operation on a function that affects the graph of the function.

Transformation of Functions

Definitions, Rules and Formulas

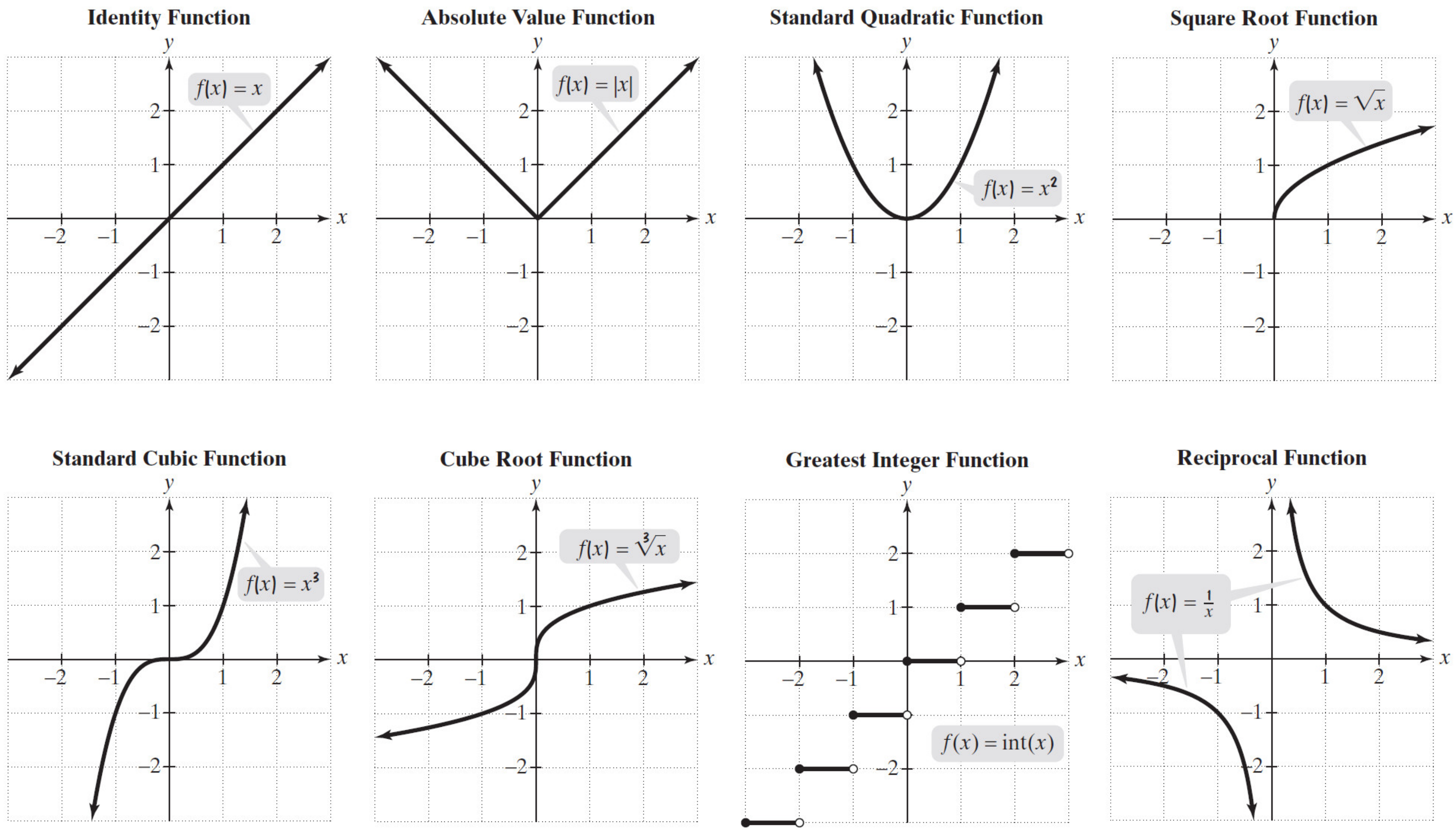
At the end of the book there is a very handy 4-pages reference to use.

The pdf file with them is posted on our website.

Make sure to keep it handy!

Transformation of Functions

ALGEBRA'S COMMON GRAPHS



Transformation of Functions

Vertical shifts

[Def] Given a function $f(x)$ a new function $g(x) = f(x) + c$, where c is a constant, is a *vertical shift of the function*.

All the output values change by c units.
If c is positive, the graph will shift up.
If c is negative, the graph will shift down.

Examples:
$f(x) + 5$ is a vertical shift 5 units up
$f(x) - 11$ is a vertical shift 11 units down

Transformation of Functions

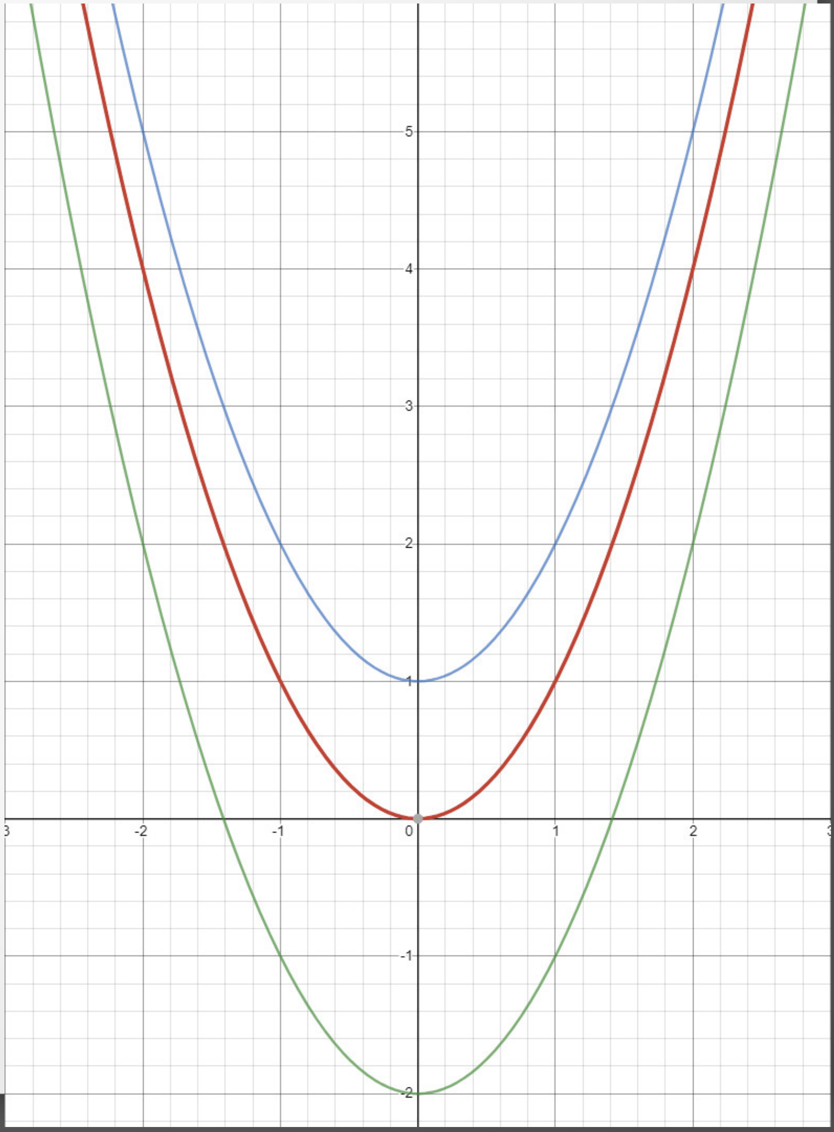
Vertical shifts

Example: consider three graphs:

$f(x) = x^2$

$g(x) = x^2 + 1$

$h(x) = x^2 - 2$



Transformation of Functions

Horizontal shifts

[Def] Given a function $f(x)$ a new function $g(x) = f(x+c)$, where c is a constant, is a *horizontal shift of the function*.

If c is positive, the graph will shift left.
If c is negative, the graph will shift right.

Examples:
$f(x+5)$ is a horizontal shift 5 units left
$f(x-11)$ is a horizontal shift 11 units right

Transformation of Functions

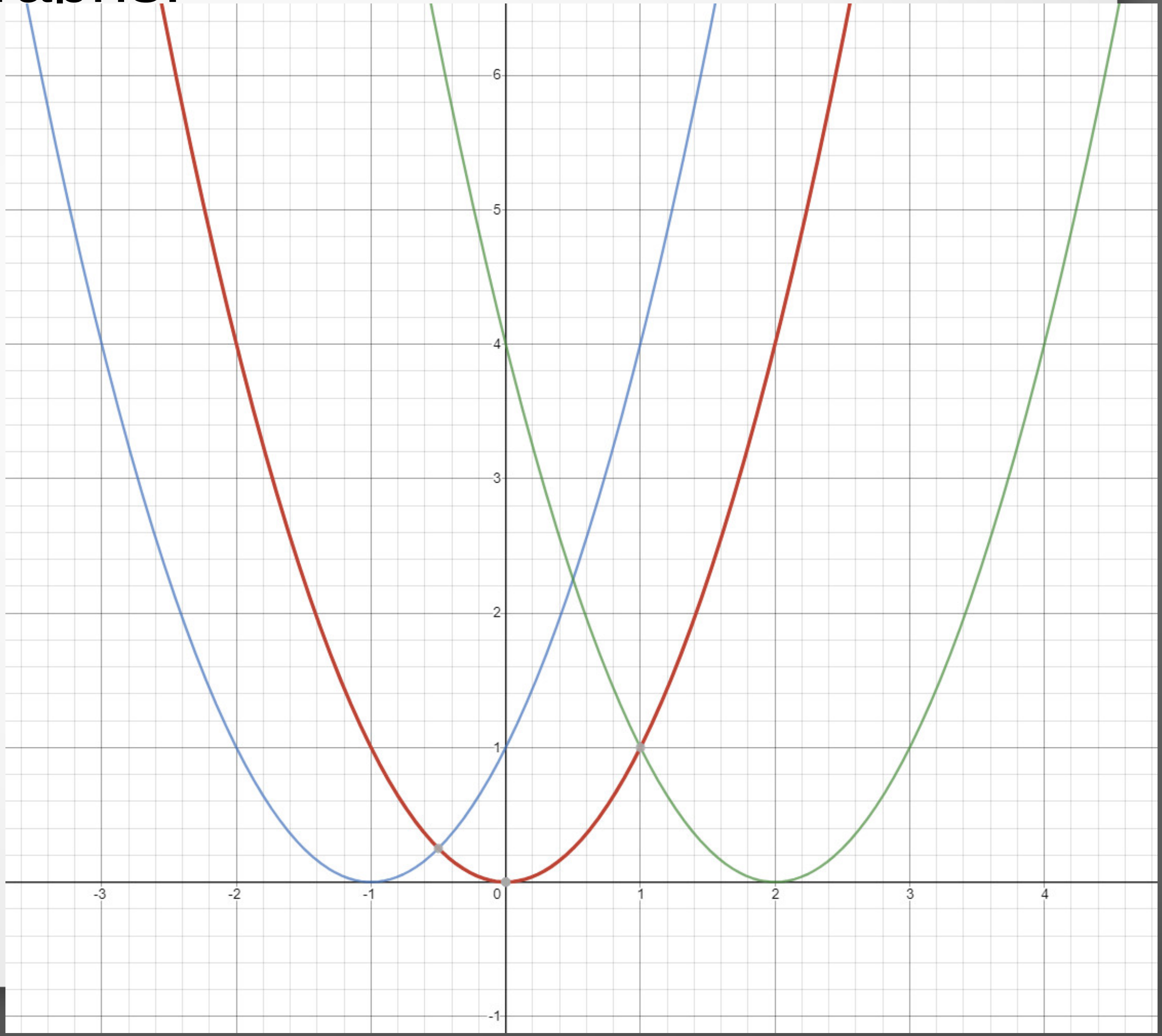
Horizontal shifts

Example: consider three graphs:

$f(x) = x^2$

$g(x) = (x+1)^2$

$h(x) = (x-2)^2$



Transformation of Functions

Combining vertical and horizontal shifts

Vertical shifts are <u>outside</u> changes that affect the <u>output</u>, <u>y-axis values</u>, and shift the function <u>up</u> or <u>down</u>.

Horizontal shifts are <u>inside</u> changes that affect the <u>input</u>, <u>x-axis values</u> and shift the function <u>left</u> or <u>right</u>.

Transformation of Functions

Combining vertical and horizontal shifts

Vertical shifts are <u>outside</u> changes that affect the <u>output</u>, <u>y-axis values</u>, and shift the function <u>up</u> or <u>down</u>.

Horizontal shifts are <u>inside</u> changes that affect the <u>input</u>, <u>x-axis values</u> and shift the function <u>left</u> or <u>right</u>.

It is recommended to “work” on the function’s expression before performing the shifts, see the example later on.

Transformation of Functions

Combining vertical and horizontal shifts

Example:
given the graph of $f(x) = x^2$, graph $h(x) = (x+7)^2-6$.

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Transformation of Functions

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horizontal shift
7 units left

vertical shift
6 units down

Transformation of Functions

Combining vertical and horizontal shifts

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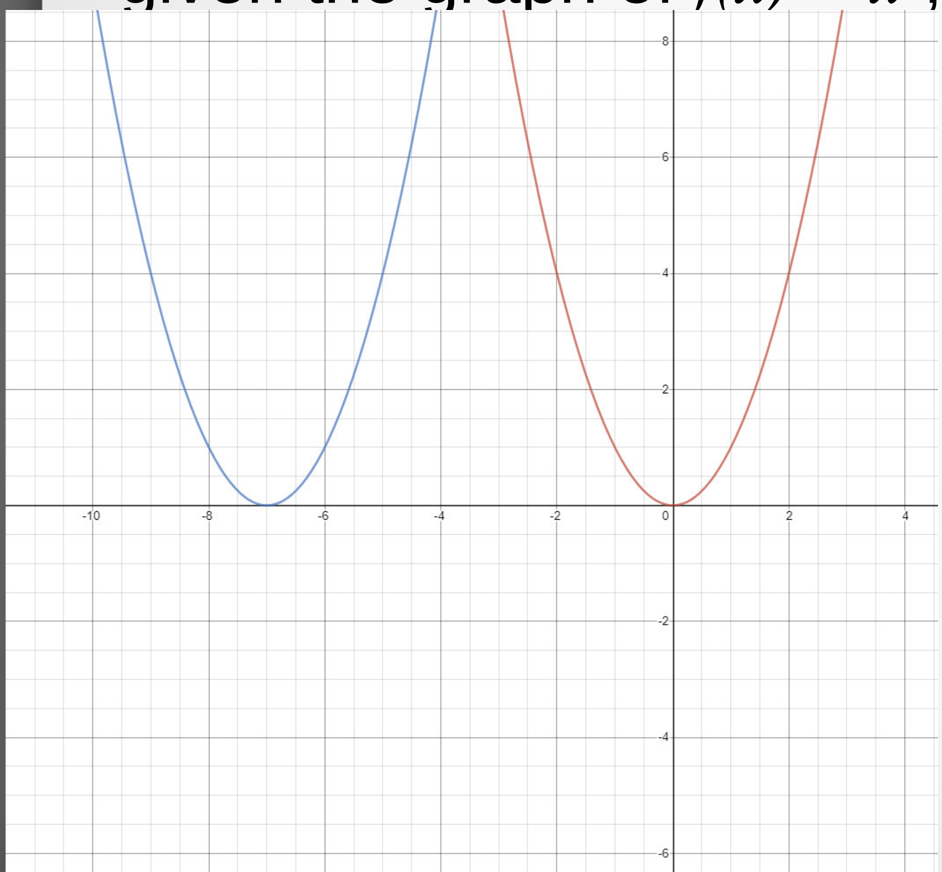

the expression for $h(x)$ in terms

horizontal shift
7 units left

vertical shift
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Transformation of Functions

Combining vertical and horizontal shifts

Example:

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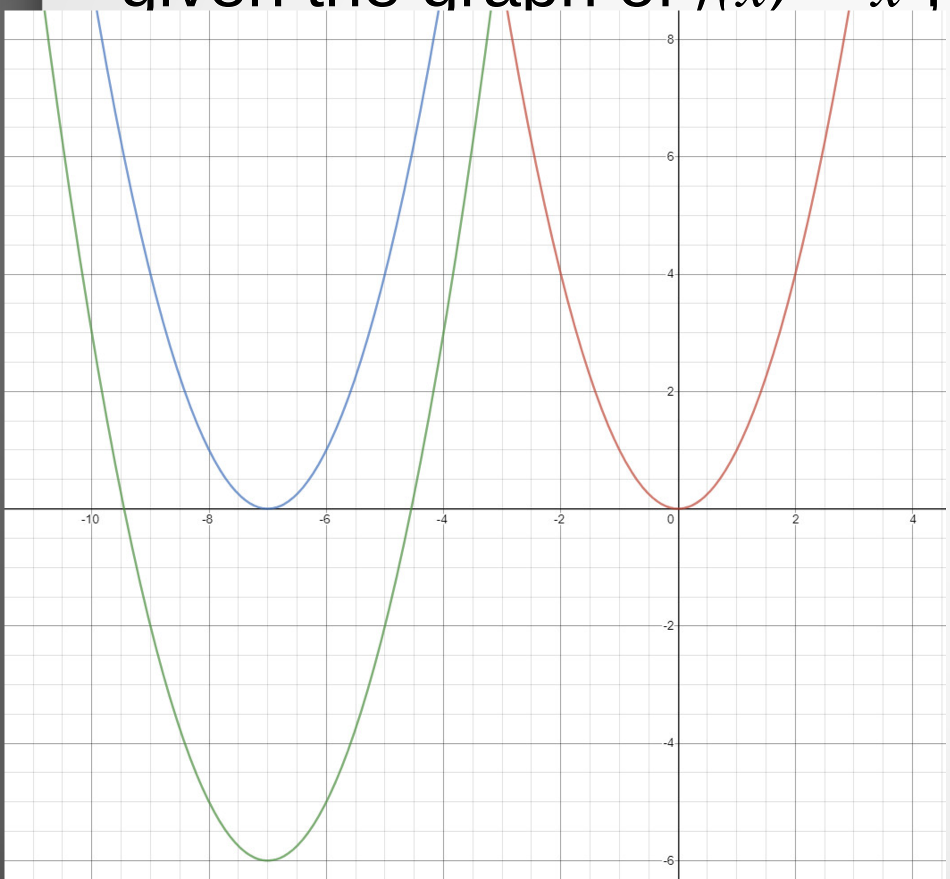


the expression for $h(x)$ in terms

horizontal shift
7 units left

vertical shift
6 units down

Transformation of Functions

Combining vertical and horizontal shifts

Example:
given the graph of $f(x) = x^2$, graph $h(x) = x^2 - 6x + 13$.

Solution:
our goal: re-write expression of $h(x)$ in terms of $f(x)$:

$h(x) = x^2 - 6x + 13 = (x-3)^2 + 4$ "completing the square"

then $h(x) = f(x-3) + 4$

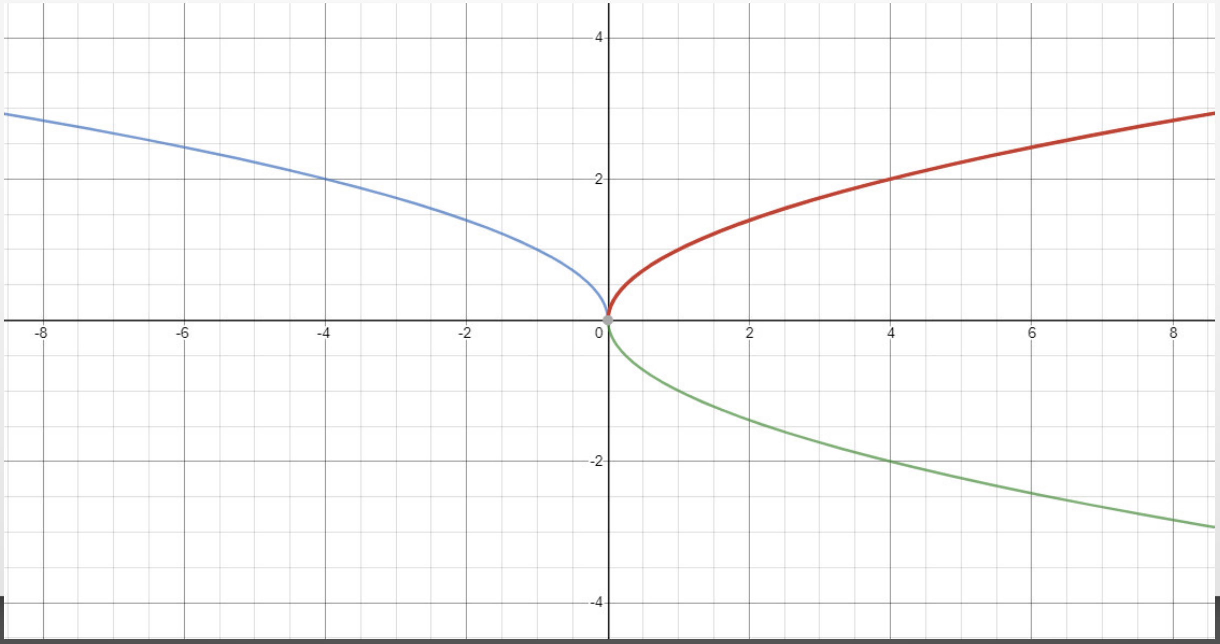
and we can conclude that we need to do:
– vertical shift 4 units up and
– horizontal shift 3 units right.

Transformation of Functions

Reflections

[Def] A *vertical reflection* reflects a graph vertically across the x-axis, while a *horizontal reflection* reflects a graph horizontally across the y-axis.

Examples: consider the graphs of $\sqrt{x}$, $\sqrt{-x}$ and $-\sqrt{x}$:



Transformation of Functions

Reflections

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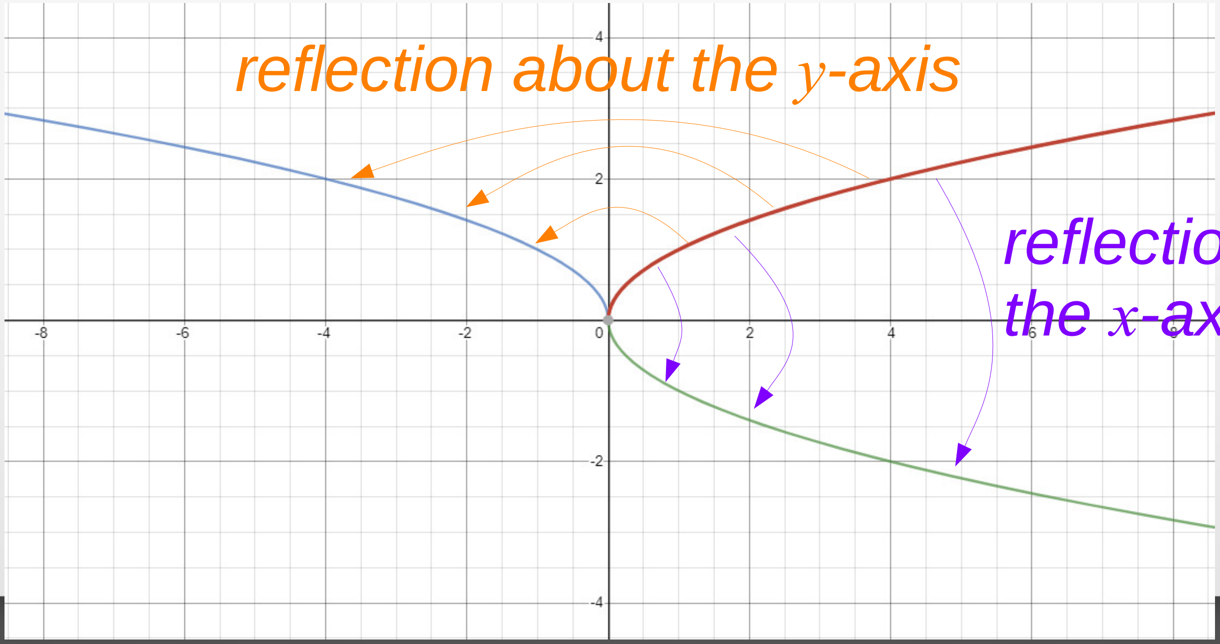
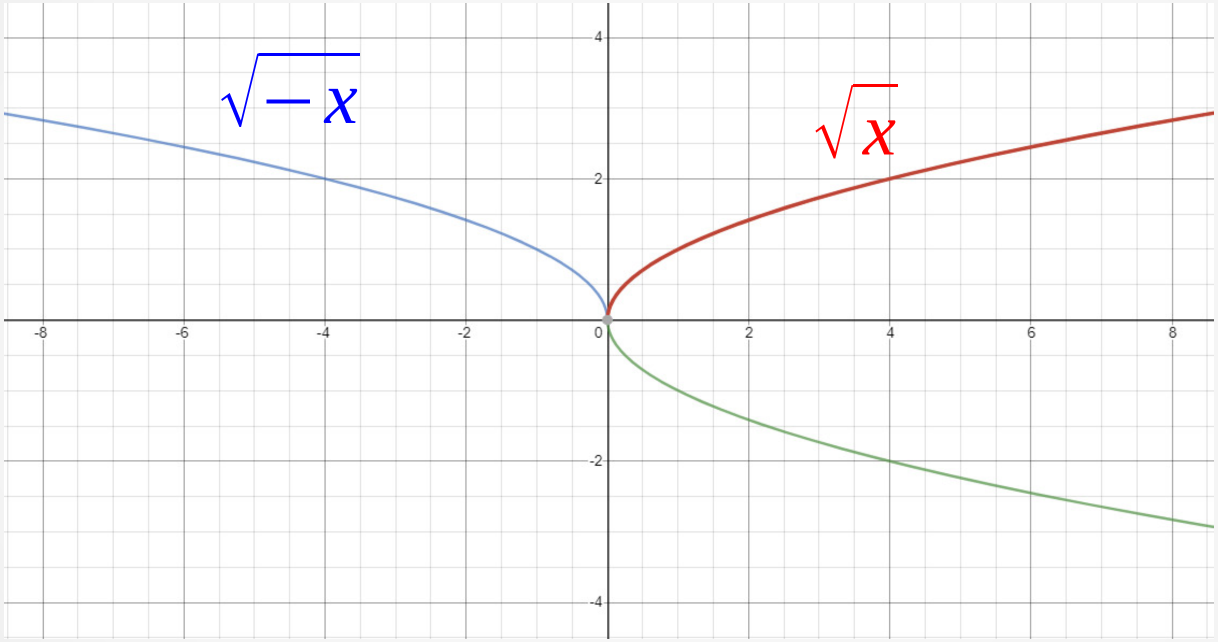


Transformation of Functions

Reflections



So, given $f(x)$,
$f(-x)$ gives us *reflection about the y-axis*, and
$-f(x)$ gives us *reflection about the x-axis*.

Transformation of Functions

Vertical stretching and shrinking/compression

[Def] Given a function $f(x)$,
$cf(x)$ is a *vertical stretching* if $c > 1$
$cf(x)$ is a *vertical shrinking/compression* if $0 < c < 1$
$cf(x)$ is a combination of *vertical stretching and vertical reflection* if $c < -1$
$cf(x)$ is a combination of *vertical shrinking and vertical reflection* if $-1 < c < 0$

Transformation of Functions

Vertical stretching and shrinking/comp

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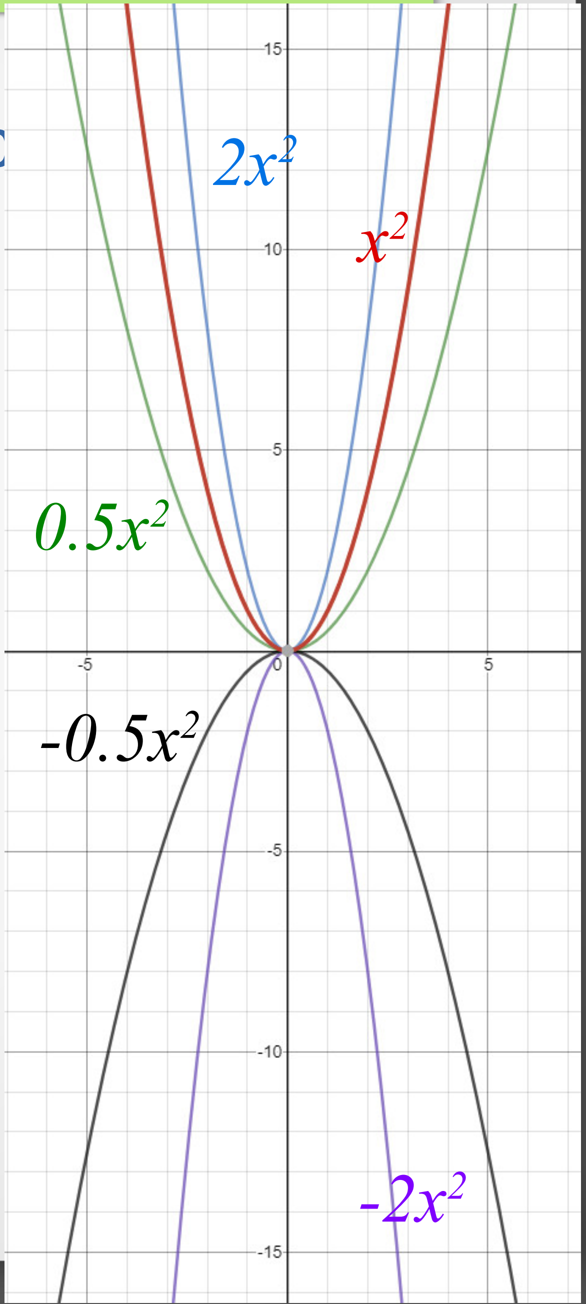

Transformation of Functions

Horizontal stretching and shrinking/compression

[Def] Given a function $f(x)$,
$f(cx)$ is a *horizontal shrinking/compression* if $c > 1$ (by $\frac{1}{c}$)
$f(cx)$ is a *horizontal stretching* if $0 < c < 1$ (by $1/c$)
$f(cx)$ is a combination of *horizontal shrinking and reflection* if $c < -1$
$f(cx)$ is a combination of *horizontal stretching and reflection* if $-1 < c < 0$

Transformation of Functions

Horizontal stretching and shrinking/compression

[Def] Given a function $f(x)$,

$f(cx)$ is a *horizontal shrinking/compression* if $c > 1$ (by $\frac{1}{c}$)

$f(cx)$ is a *horizontal stretching* if $0 <$

$f(cx)$ is a combination of
horizontal shrinking and reflection if $c < -1$

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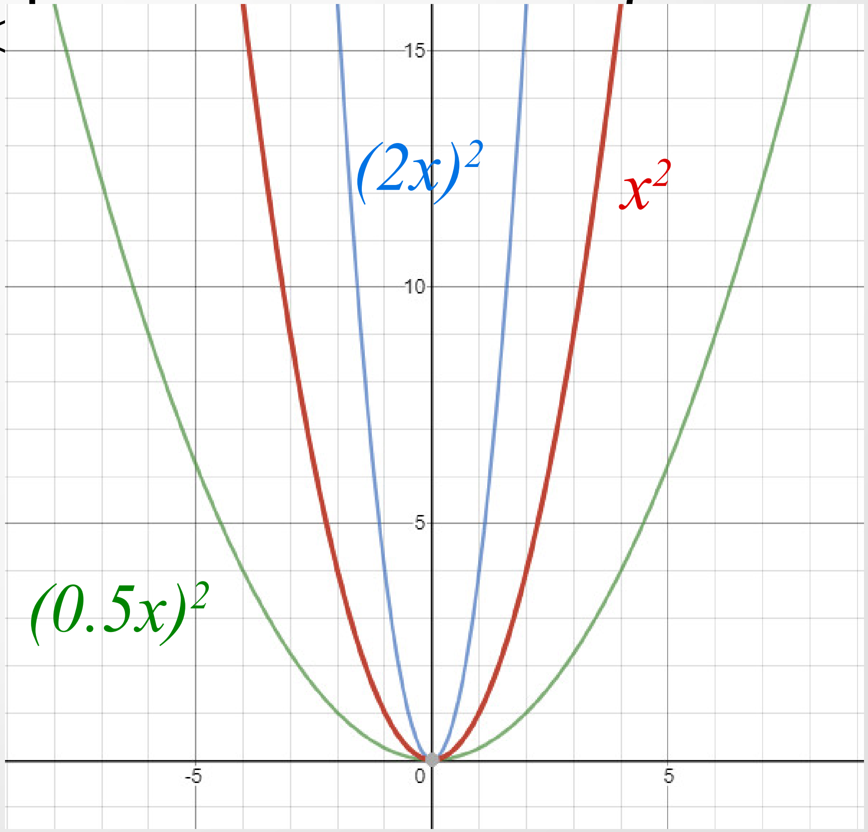


Transformation of Functions

Combining Transformations

When <u>combining *vertical transformations*</u> written in the form $af(x)+k$, first *vertically stretch/shrink* by a and then *vertically shift* by k.

When <u>combining *horizontal transformations*</u> written in the form $f(ax+k)$, first *horizontally shift* by k and then *horizontally stretch/shrink* by $1/a$.

When <u>combining *horizontal transformations*</u> written in the form $f(a(x+k))$, first *horizontally stretch/shrink* by $1/a$ and then *horizontally shift* by k.

Horizontal and vertical transformations are independent.
It does not matter whether horiz. or vert. transformations are performed first.

Transformation of Functions

A suggested sequence of Transformations

- Horizontal shifting
- Stretching or shrinking (vertical / horizontal)
- Reflecting
- Vertical shifting

Transformation of Functions

Example 1: get the graph of $(2x-3)^2$ by using the graph of x^2.

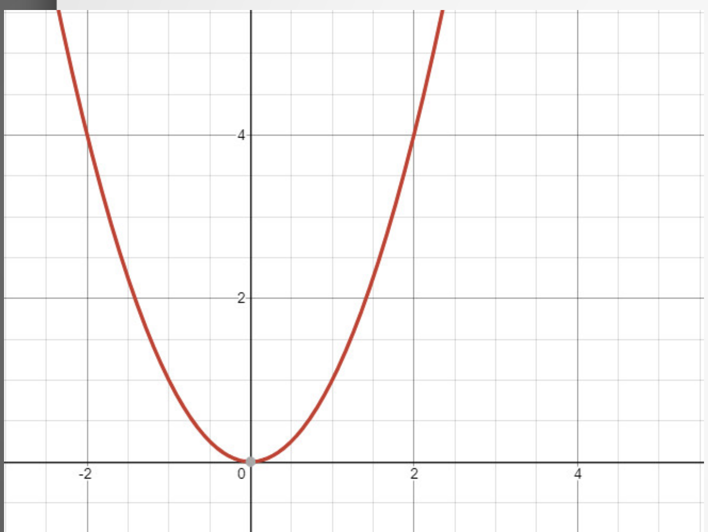
Transformation of Functions

Example 1: get the graph of $(2x\text{-}3)^2$ by using the graph of x^2.

Let's use this:
When <u>combining *horizontal transformations*</u> written in the form $f(ax+k)$, first *horizontally shift* by k and then *horizontally stretch/shrink* by $1/a$.

Transformation of Functions

Example 1: get the graph of $(2x-3)^2$ by using the graph of x^2.



horizontal shift
3 units right

Transformation of Functions

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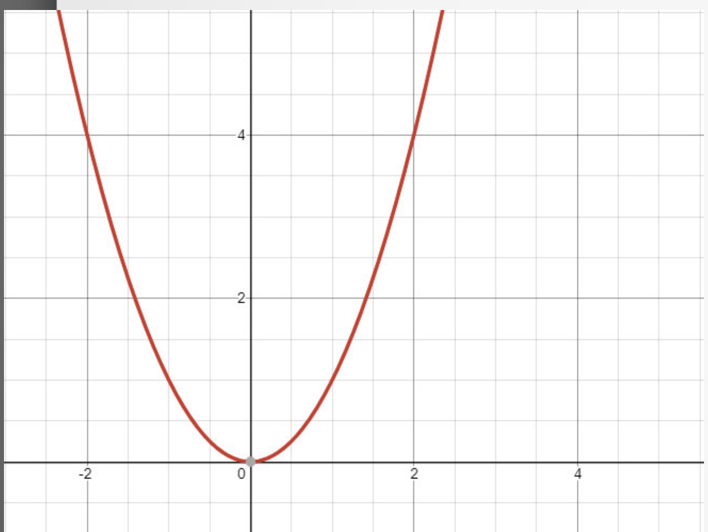
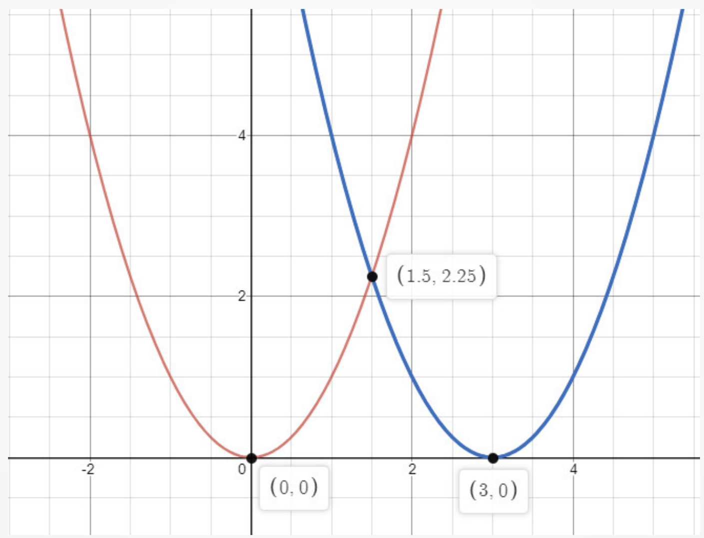
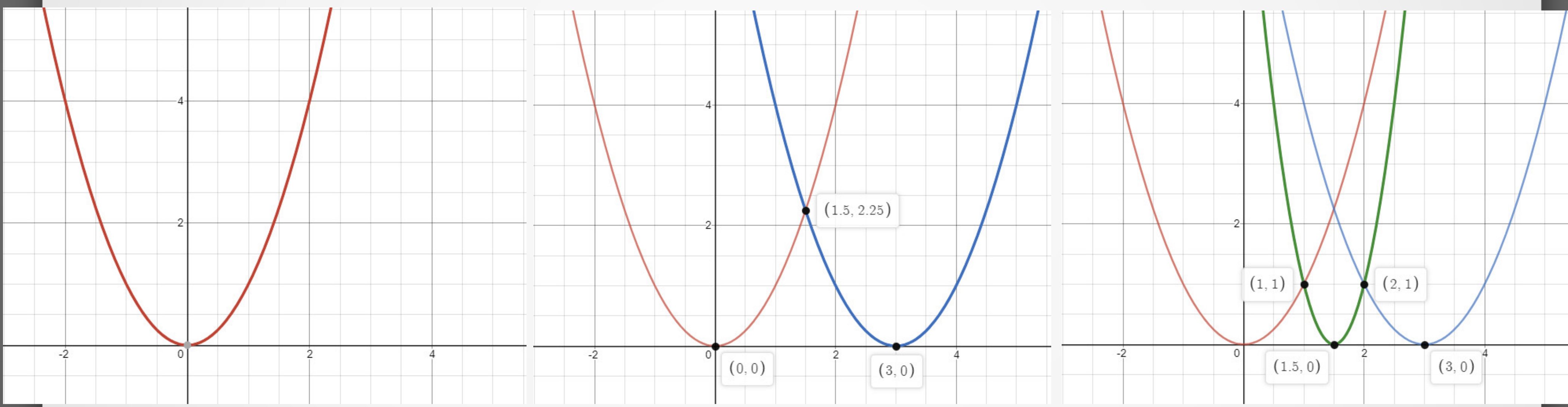




horizontal shift
3 units right

Transformation of Functions

Example 1: get the graph of $(2x-3)^2$ by using the graph of x^2.

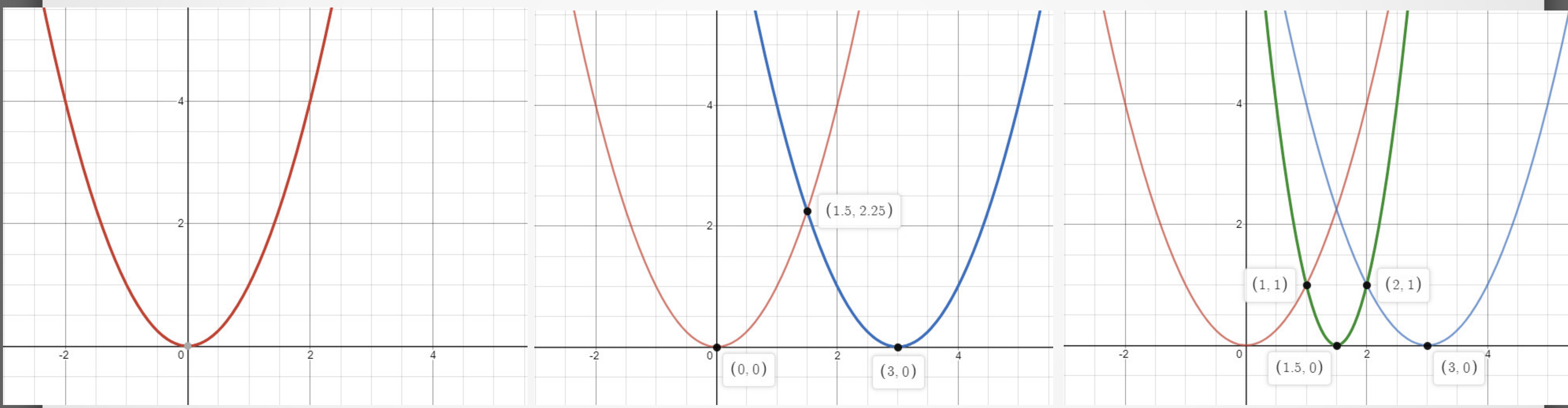


horizontal shift
3 units right

horizontal shrinking
by 1/2

Transformation of Functions

Example 1: get the graph of $(2x-3)^2$ by using the graph of x^2.

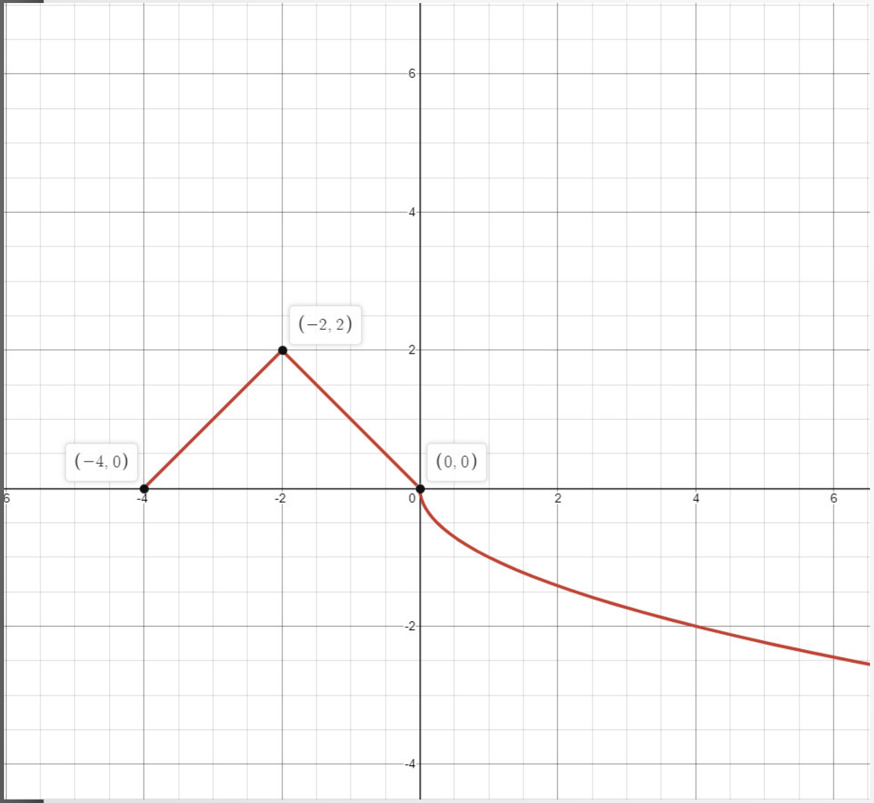


Exercise: get the graph of $(2(x-1.5))^2$ by using the graph of x^2.

Note that the resulting graph should be the same!

Transformation of Functions

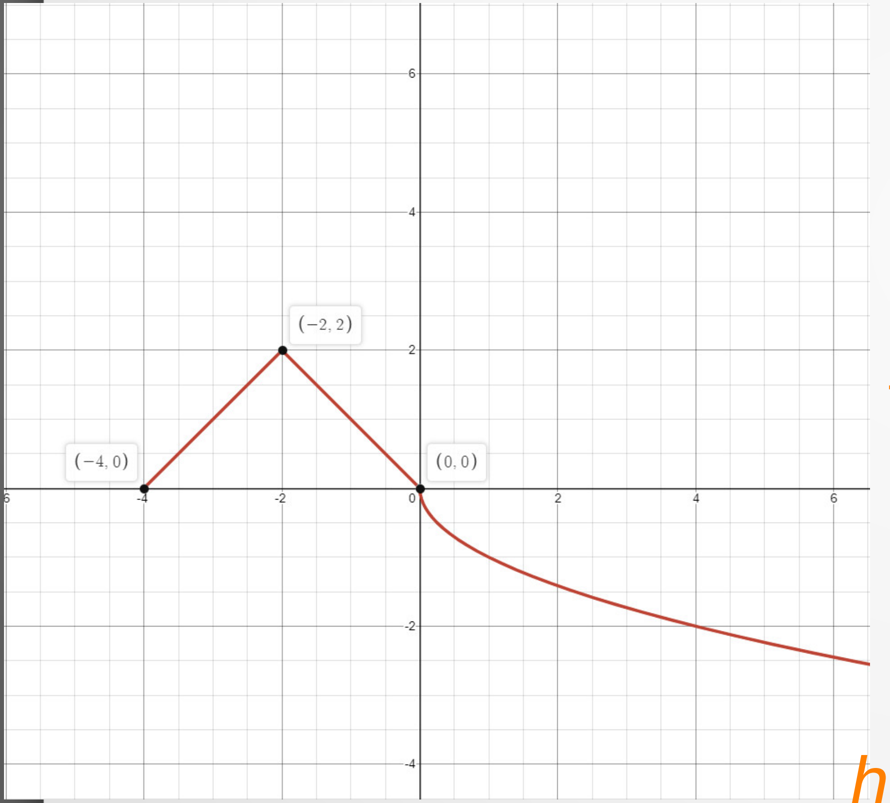
Example 2: Use the given graph of $f(x)$ to graph $g(x) = -3f(x-2) + 4$.

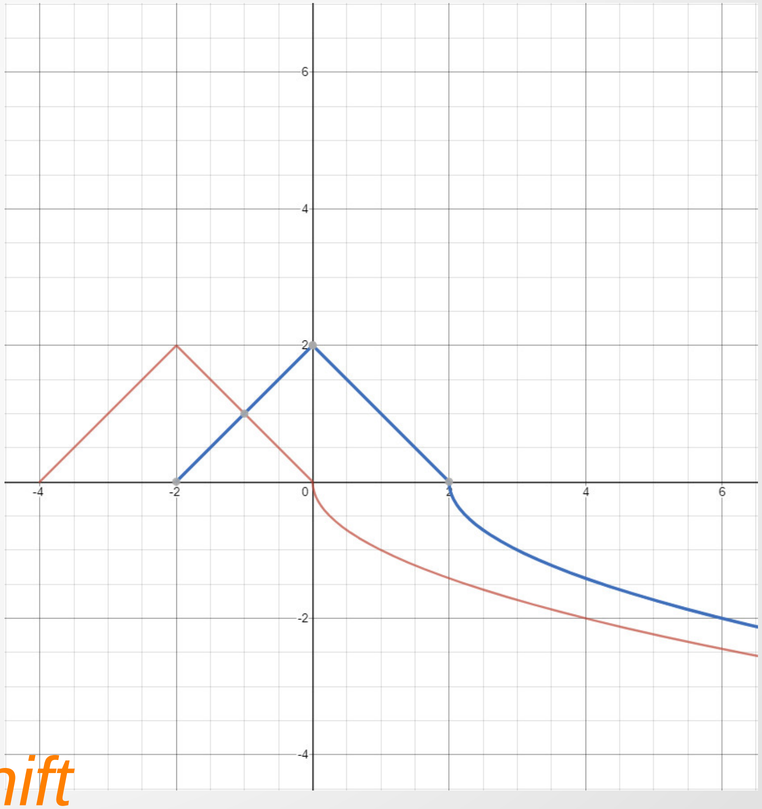


Transformation of Functions

Example 2: Use the given graph of $f(x)$ to graph

$$g(x) = -3f(x-2) + 4.$$

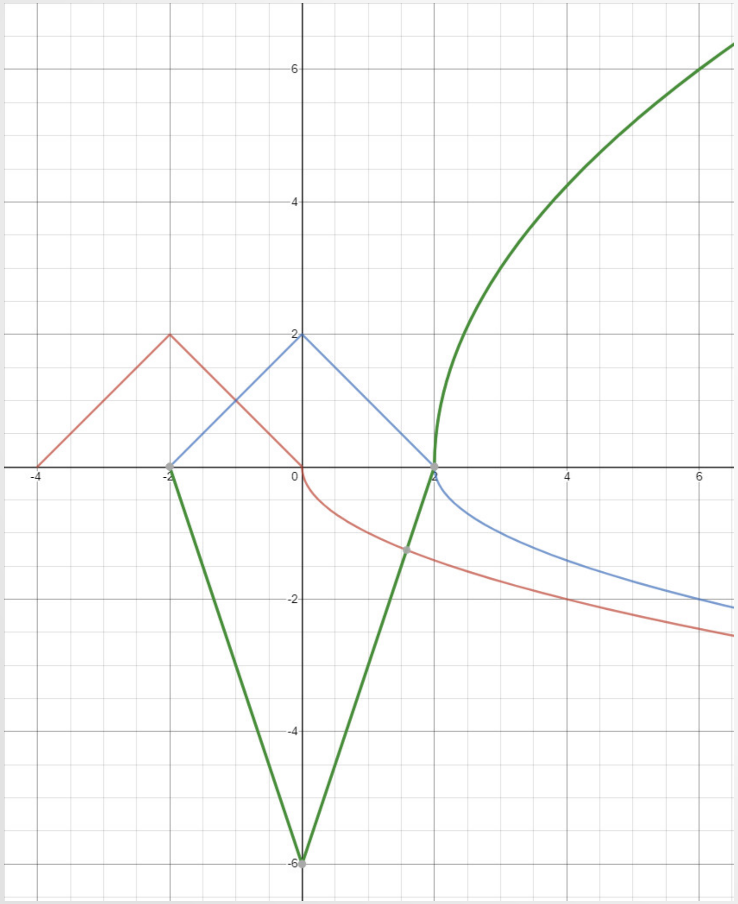




horizontal shift
2 units right

Transformation of Functions

Example 2: Use the given graph of $f(x)$ to graph
$g(x) = -3f(x-2) + 4$.



vertical stretching (by 3) and reflection about the x-axis

Transformation of Functions

Example 2: Use the given graph of $f(x)$ to graph

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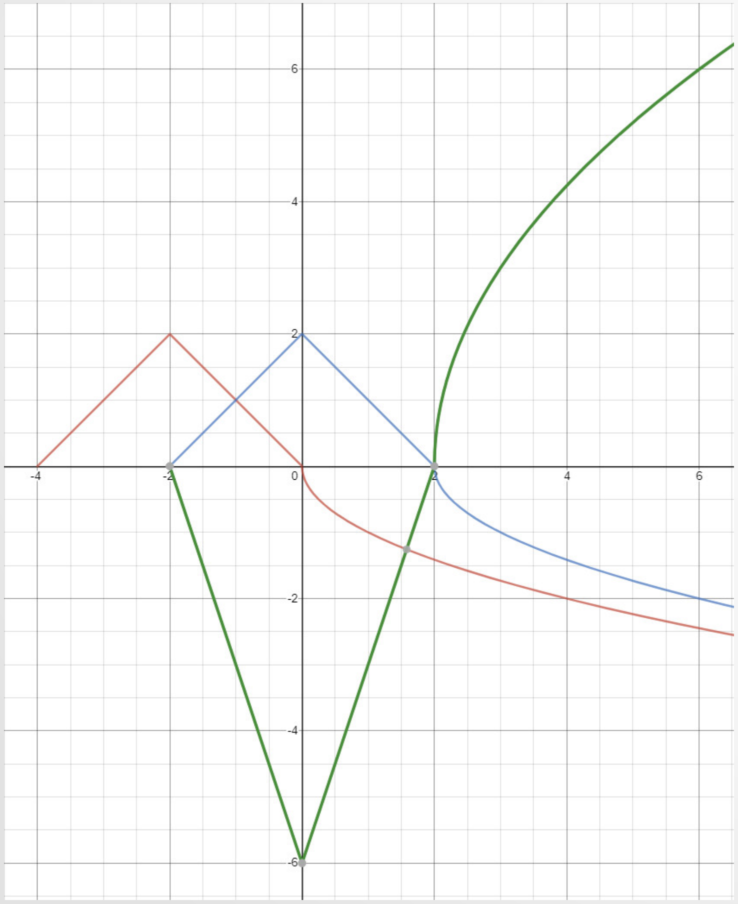

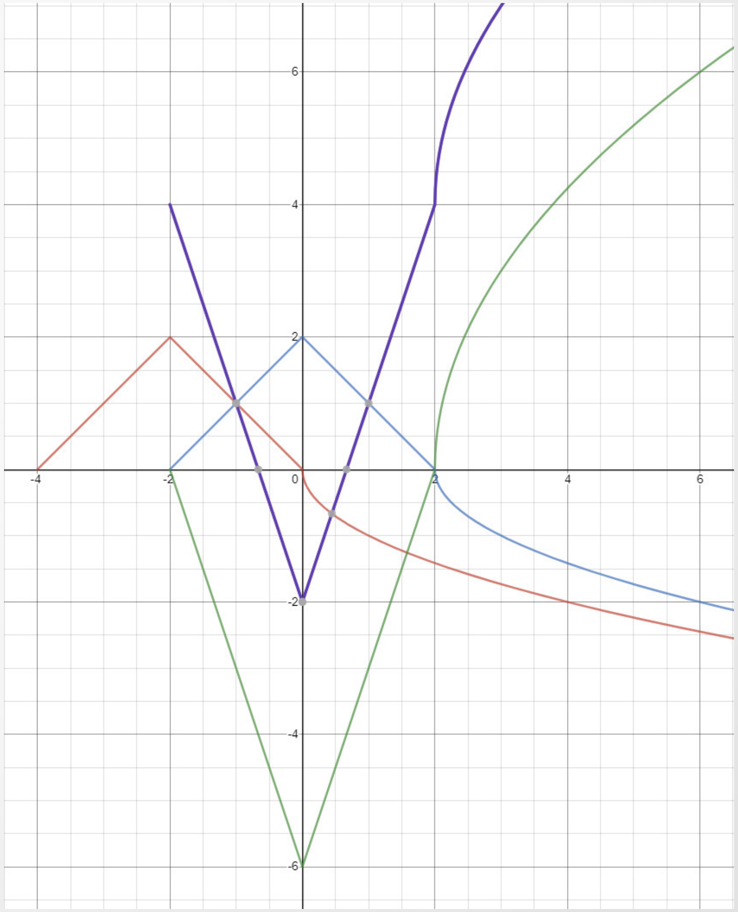

vertical shift
4 units up

Homework assignment

1) zyBooks: *review* Section 1.6 *Transformation of Functions*
read Section 1.7 *Inverse Functions*

or

Textbook: *review* Section 1.6 *Transformations of Functions*
read Section 1.8 *Inverse Functions*.

2) We will have Quiz 4 based on today's topics in the beginning of our next meeting.

3) WeBWorK: HW 4 (due date is in one week)