8

Geometry

8.9 Fractal Geometry

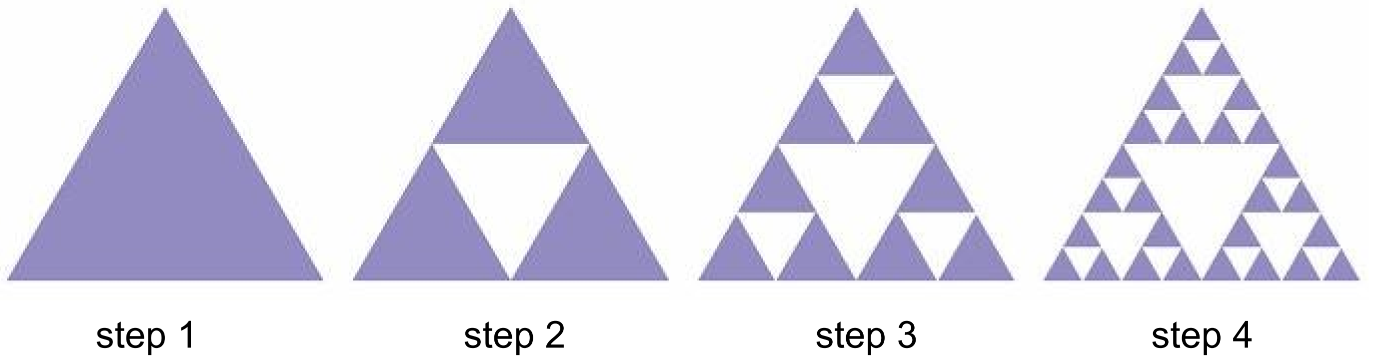
Objectives

- Understand how a recursive process can be used to generate a fractal
- See how self-similarity occurs in both the real world and fractal geometry
- Be able to comprehend and compute a fractal’s fractional dimension
- Grasp that some real-world objects have fractional dimensions

The Sierpinski Gasket

The Sierpinski Gasket

The **Sierpinski gasket** is one of the older examples of fractal geometry. It was first studied by Polish mathematician Waclaw Sierpinski in 1915. To understand this shape, we will create one. Take the following steps, which are illustrated in Figure 8.144.



Creating a Sierpinski gasket.

Figure 8.144

The Sierpinski Gasket

1. Draw a triangle and fill in its interior.

2. Put a hole in the triangle:

- Find the midpoints of the sides of the triangle.
- Connect the midpoints and form a new triangle.
- Form a hole by removing the new triangle.

This leaves three filled in triangles, each a smaller version of the original triangle from step 1.

3. Put a hole in each of the three triangles from step 2 by applying the procedure from step 2 to each triangle.

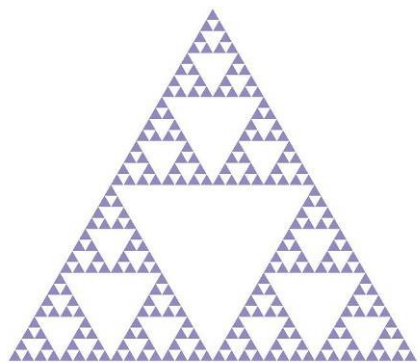
The Sierpinski Gasket

4. Put a hole in each of the triangles from step 3 by applying the procedure from step 2 to each triangle.

5. Put a hole in each of the triangles from the previous step by applying the procedure from step 2 to each triangle. Then do it again and again.

The Sierpinski Gasket

The Sierpinski gasket (Figure 8.145) is the result of continuing the process described above through steps 6 and 7 and 8 and beyond without stopping.



Step 5 of a Sierpinski gasket.

Figure 8.145

The Sierpinski Gasket

After several steps, it becomes impossible to visually tell the difference between the result of one step and the result of the next step.

That is, you probably couldn’t tell the difference between the result of stopping at step 10 and the result of stopping at step 11 unless you used a magnifying glass.

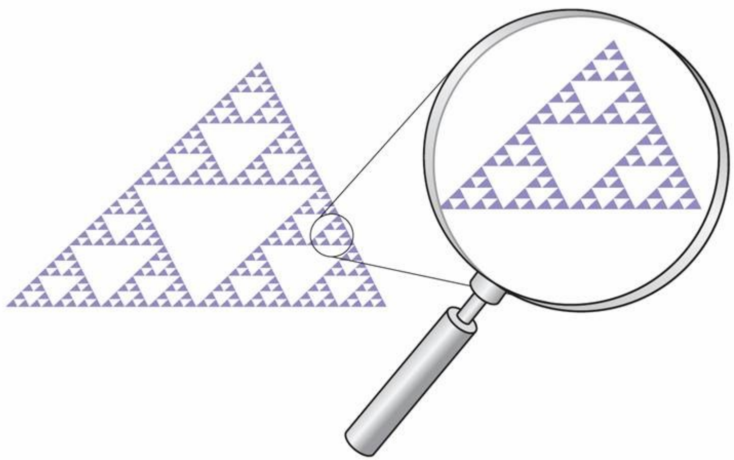
The Sierpinski Gasket

However, neither of these shapes would be a Sierpinski gasket; the only way to get a Sierpinski gasket is to continue the process without stopping, forming more and more ever-tinier triangles, each with holes and even tinier triangles inside it.

Self-Similarity

Self-Similarity

The enlarged portion would look exactly the same as the non enlarged portion, as shown in Figure 8.146. This is called *self-similarity*.



The self-similarity of a Sierpinski gasket.

Figure 8.146

Self-Similarity

A shape has **self-similarity** if parts of that shape appear within itself at different scales. Fractal geometry utilizes shapes that have self-similarity.

Some shapes have **exact self-similarity** (in which the enlarged portion is exactly the same as the nonenlarged portion), and some shapes have **approximate self-similarity.**

Self-Similarity

Approximate self-similarity is a common feature of shapes that exist in nature. For example, trees and lightning can have a self-similar structure.

The fact that fractal geometry utilizes self-similarity is one reason why fractal geometry generates shapes that could exist in the real world.

Recursive Processes

Recursive Processes

A process in which a procedure is applied and the same procedure is applied to the result, over and over again, is called a **recursive process.**

Two of the key elements of fractal geometry are self-similarity and recursive processes.

The term **fractal,** can be described as a mathematical object whose form is extremely irregular and/or fragmented at all scales.

The Skewed Sierpinski Gasket

The Skewed Sierpinski Gasket

The **skewed Sierpinski gasket** is a simple variation of the Sierpinski gasket.

1. Draw a triangle and fill in its interior.

2. Put a twisted hole in the triangle: (See Figure 8.150.)

The midpoints and nearby points in step 2

Finishing step 2

Creating a skewed Sierpinski gasket.

Figure 8.150

The Skewed Sierpinski Gasket

- Find the midpoints of the sides of the triangle (as before).
- For each midpoint, find a nearby point (this is the new twist).
- Connect the nearby points and the vertices of the original triangle.
- Form a twisted hole by removing the center triangle.

3. Put a twisted hole in each of the three triangles from step 2 by applying the procedure from step 2 to each triangle.

The Skewed Sierpinski Gasket

4. Put a twisted hole in each of the triangles from step 3 by applying the procedure from step 2 to each triangle.

5. Put a twisted hole in each of the triangles from the previous step by applying the procedure from step 2 to each triangle. Continue this process indefinitely.

The Skewed Sierpinski Gasket

Figure 8.151 shows a skewed Sierpinski gasket. Surprisingly, this fractal looks somewhat like a mountain! It has both the shape and the texture of a real mountain. Fractal geometry generates shapes that could exist in the real world.

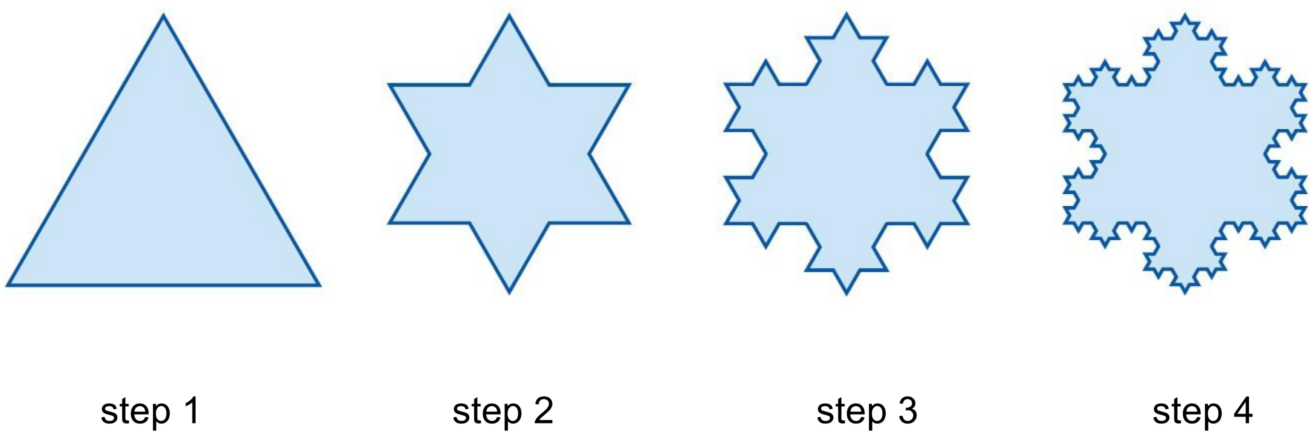
A skewed Sierpinski gasket and a real mountain.

Figure 8.151

The Koch Snowflake

The Koch Snowflake

The **Koch snowflake** is another example of fractal geometry. It was first studied by Swedish mathematician Helge von Koch in 1904. To create this shape, follow these steps, which are illustrated in Figure 8.156.



Creating a Koch snowflake.

Figure 8.156

The Koch Snowflake

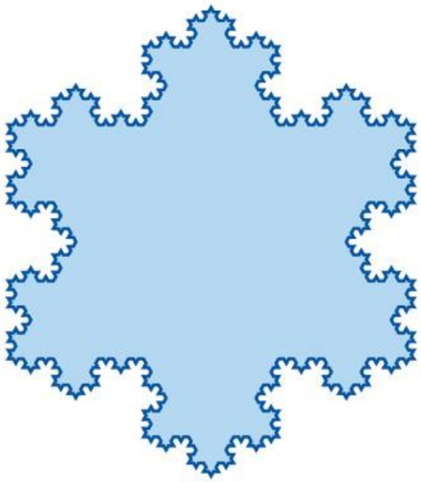
1. Draw an equilateral triangle (that is, a triangle whose sides are equal and whose angles are equal).

2. In the middle of each side of the equilateral triangle, attach a smaller equilateral triangle (pointing outwards). The smaller equilateral triangle’s sides should be one third as long as the original larger equilateral triangle’s sides. This forms a Star of David, with 12 sides.

3. In the middle of each side of the Star of David, attach an equilateral triangle (pointing outward). This equilateral triangle’s sides should be one third as long as the Star of David’s sides.

The Koch Snowflake

4. Alter the result of the previous step by attaching an equilateral triangle to the middle of each side. Continue this process indefinitely. The resulting curve, shown in Figure 8.157, is called a Koch snowflake. Notice its similarity to a real snowflake, such as that in Figure 8.158.



A Koch snowflake.

Figure 8.157

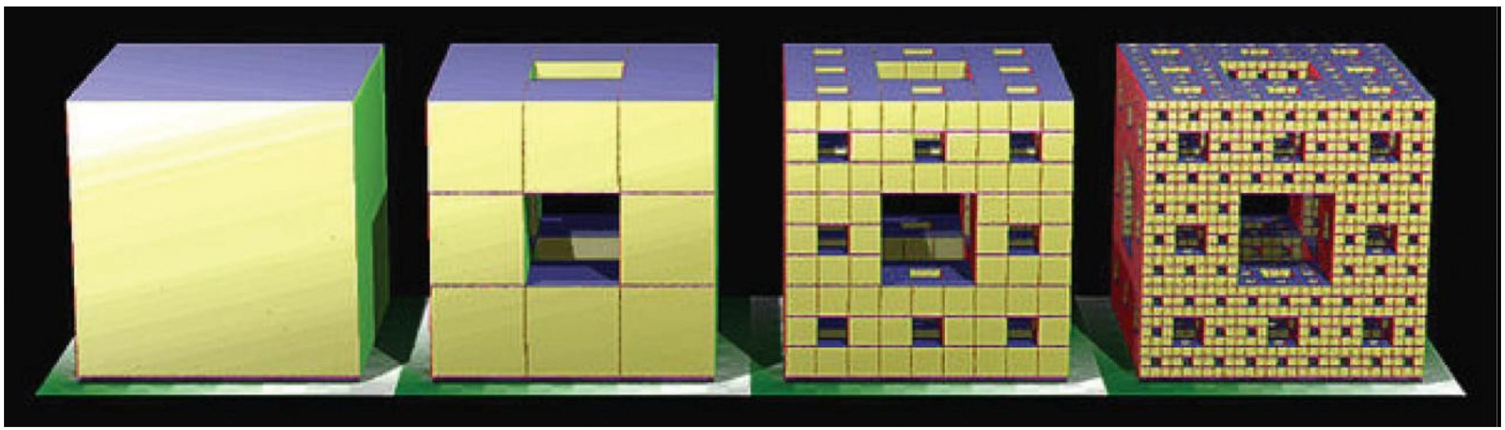
Notice the similarity between the Koch snowflake and the real snowflake shown here.

Figure 8.158

The Menger Sponge

The Menger Sponge

The **Menger sponge** is closely related to the Sierpinski gasket. To create this shape, follow these steps, which are illustrated in Figure 8.159.



step 1 step 2 step 3 step 4

Creating a Menger sponge.

Figure 8.159

The Menger Sponge

1. Draw a cube.

2. Remove some of the cube:

- Subdivide each of the cube's faces into nine equal squares. This subdivides the cube itself into 27 small cubes (nine in the front, nine in the middle, and nine in the back).
- Remove the small cube at the center of each face. There are six such small cubes.
- Remove the small cube at the very center of the original cube.

The Menger Sponge

3. A number of small cubes remain at the end of step 2. Remove some of each of these cubes by applying the procedure from step 2 to them.

4. A number of cubes remain at the end of the previous step. Remove some of each of these cubes by applying the procedure from step 2 to them. Continue this process

indefinitely. The resulting shape is called a Menger sponge.

Some Applications of Fractals

Some Applications of Fractals

Biologists use *diffusion fractals* to analyze how bacteria cultures grow.

Biologists also use *l-systems,* a method of generating fractals, to study the structures of plants.

Chemists use *strange attractors,* a type of fractal, to study chaotic behavior in chemical reactions.

Anatomists use *fractal canopies* to study the structure of lungs.

Some Applications of Fractals

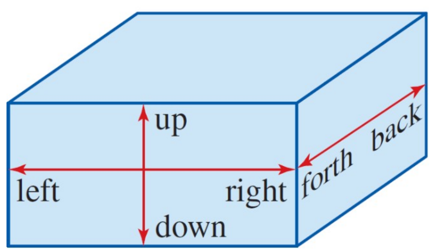
Computer scientists use *fractal image compression* to squeeze 7,000 photographs onto the Microsoft Encarta encyclopedia CD.

Movie makers use fractals to create special effects in movies.

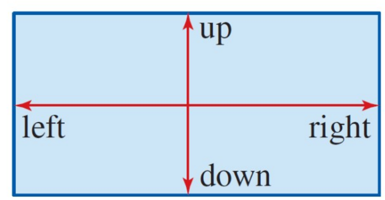
Dimension

Dimension

A box has three dimensions: up/down, back/forth, and left/right. A rectangle has two dimensions: up/down and left/right. A line has one dimension: left/right. A circle (one that's not filled in) also has one dimension: clockwise/counterclockwise.

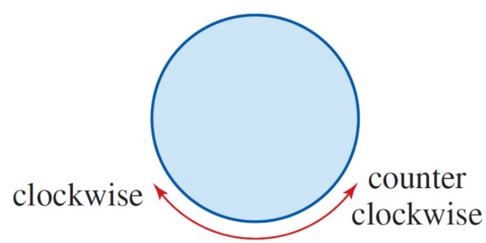


A box's 3 dimensions



A rectangle's 2 dimensions

A line's 1 dimension



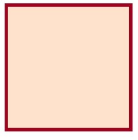
A circle's 1 dimension

The Dimension of a Fractal

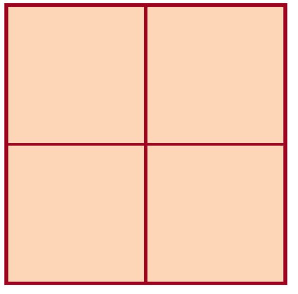
The Dimension of a Fractal

The dimension of a fractal is certainly less obvious than the dimension of a box, a rectangle, or a line. Because of this, it must be determined with a new approach. We'll start by applying this new approach to a square.

If we take a square and double both its length and its width, we get four new squares, each equal to the original square.



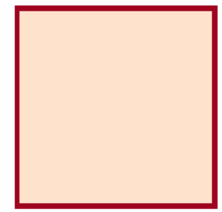
The original square



Double its length and width and get 4 new squares, each equal to the original square.

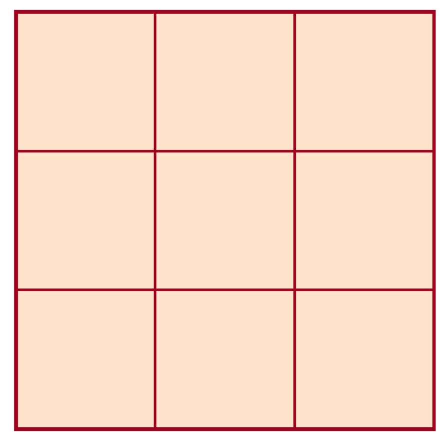
The Dimension of a Fractal

We say that the scale factor is 2, because we multiplied the length and width by 2.



The original square

What happens if we apply a scale factor of 3 to the original square? If we multiply both the length and the width by 3, we get nine new squares, each equal to the original square.

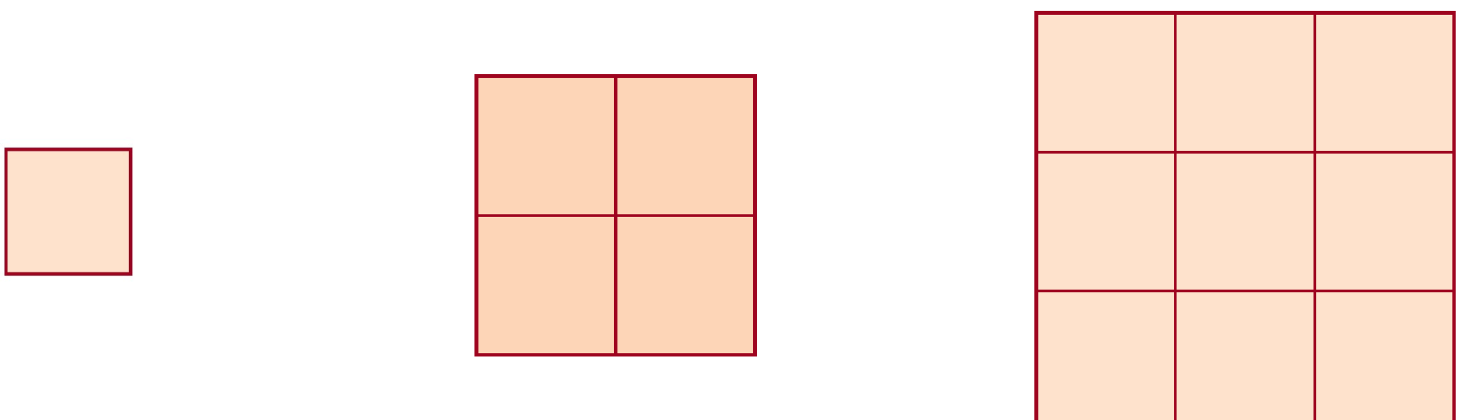


Triple its length and width and get 9 new squares, each equal to the original square.

The Dimension of a Fractal

In each case, the scale factor, raised to the second power, gives the number of new squares:

- When the scale factor is 2, we get $2^2 = 4$ new squares.
- When the scale factor is 3, we get $3^2 = 9$ new squares.



The Dimension of a Fractal

In each case, the scale factor, raised to the second power, gives the number of new squares:

- When the scale factor is 2, we get $2^2 = 4$ new squares.
- When the scale factor is 3, we get $3^2 = 9$ new squares.

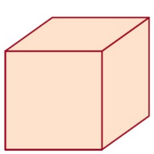
What happens with cubes?

The Dimension of a Fractal

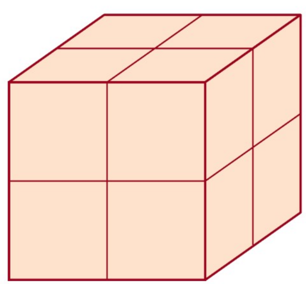
In each case, the scale factor, raised to the second power, gives the number of new squares:

- When the scale factor is 2, we get $2^2 = 4$ new squares.
- When the scale factor is 3, we get $3^2 = 9$ new squares.

What happens with cubes? If we apply a scale factor of 2 to a cube, then we get 8 new cubes, each equal to the original cube.



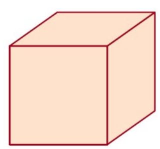
The original cube



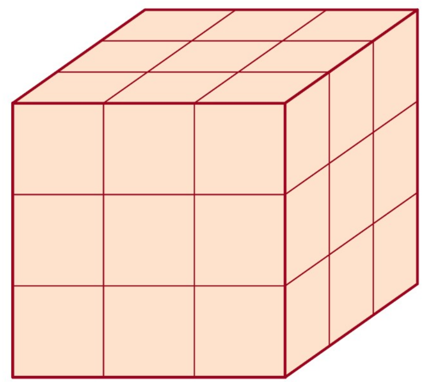
Apply a scale factor of 2 and get 8 new cubes, each equal to the original cube.

The Dimension of a Fractal

If we apply a scale factor of 3 to a cube, we get 27 new cubes, each equal in size to the original cube.



The original cube



Apply a scale factor of 3 and get 27 new cubes, each equal to the original cube.

The Dimension of a Fractal

With the square, the scale factor raised to the *second* power gives the number of new squares. But with the cube, the scale factor raised to the *third* power gives the number of new cubes:

- When the scale factor is 2, we get $2^3 = 8$ new cubes.
- When the scale factor is 3, we get $3^3 = 27$ new cubes.

Clearly, there is a relationship between the scale factor, the number of new objects, and the dimension.

The Dimension of a Fractal

The scale factor, raised to the power of the dimension, equals the number of new objects. It turns out that this relationship can be applied to fractals just as well as it can be applied to squares and cubes.

Dimension Formula

The dimension of an object is the number d that satisfies the equation

$$s^d = n$$

where s is the scale factor and n is the number of new objects that result, where each new object is equal in size to the original object.

Example 1 – *A Fractal's Dimension*

$s^d = n$

Let's find the dimension of a Koch snowflake.

Example 1 – *A Fractal's Dimension*

Let's find the dimension of a Koch snowflake.

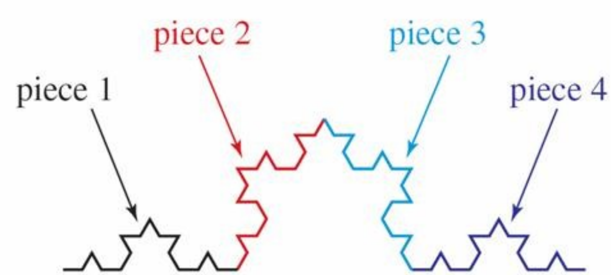
$s^d = n$

Solution:

Look at one piece of a Koch snowflake:

if we apply a scale factor of s = 3 to this piece, we get a larger piece of a Koch snowflake. This larger piece has n = 4 parts, each of which is equal in size to the original piece.

The original piece



Apply a scale factor of 3 and get 4 new pieces, each equal to the original piece.

Figure 8.164

Example 1 – *Solution*

cont'd

$s^d = n$

This means that the dimension of the piece of a Koch snowflake is the number d that solves the equation $3^d = 4$:

$s^d = n$ the dimension formula

$3^d = 4$ substituting

Example 1 – *Solution*

cont'd

$s^d = n$

This means that the dimension of the piece of a Koch snowflake is the number *d* that solves the equation $3^d = 4$:

$s^d = n$ the dimension formula

$3^d = 4$ substituting

What is d?

Example 1 – *Solution*

cont'd

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This means that the dimension of the piece of a Koch snowflake is the number d that solves the equation $3^d = 4$:

$s^d = n$ the dimension formula

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What is d?

Clearly, $d \neq 1$, because $3^1 = 3 \neq 4$.

Example 1 – *Solution*

cont'd

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What is d?

Clearly, $d \neq 1$, because $3^1 = 3 \neq 4$.

Also, $d \neq 2$, because $3^2 = 9 \neq 4$.

Example 1 – *Solution*

cont'd

$s^d = n$

This means that the dimension of the piece of a Koch snowflake is the number d that solves the equation $3^d = 4$:

$s^d = n$ the dimension formula

$3^d = 4$ substituting

What is d?

Clearly, $d \neq 1$, because $3^1 = 3 \neq 4$.

Also, $d \neq 2$, because $3^2 = 9 \neq 4$.

So d is a number between 1 and 2.

We'll approximate d to the nearest tenth, using a trial and error process.

Example 1 – *Solution*

cont'd

$s^d = n$

This means that the dimension of the piece of a Koch snowflake is the number *d* that solves the equation $3^d = 4$:

$s^d = n$ the dimension formula

$3^d = 4$ substituting

What is d?

Clearly, $d \neq 1$, because $3^1 = 3 \neq 4$.

Also, $d \neq 2$, because $3^2 = 9 \neq 4$.

So *d* is a number between 1 and 2.

$3^{1.2} \approx 3.737$ $3^{1.3} \approx 4.171$ $3^{1.25} \approx 3.948$

Since we are approximating to the nearest tenth, our answer is $d \approx 1.3$

Example 1 – *Solution*

cont'd

So the dimension of a Koch snowflake is a little bigger than the dimension of a circle. The Koch snowflake has dimension $d \approx 1.3$, while the circle has dimension $d = 1$.

Example 1 – *Solution*

cont'd

The little tiny crinkles do make the snowflake somewhat thicker than a circle. And this extra thickness makes the snowflake's dimension somewhat bigger than that of a circle. But it's quite strange that a fractal can have a dimension that is not a whole number.

Some Applications of Fractal Dimension

Some Applications of Fractal Dimension

A country's coastline has self-similarity. If you took two photographs of an uninhabited coastline, one from an elevation of 1,000 feet and the other from an elevation of 5,000 feet, you would have difficulty determining which photo was which.

This self-similarity means that fractal geometry can be used to analyze the complexity of coastlines. In fact, geographers classify coastlines according to their fractal dimension.

South Africa's coastline is quite smooth, with a fractal dimension close to 1.

Some Applications of Fractal Dimension

On the other hand, Norway's coastline is heavily indented by glacier-carved fjords; its fractal dimension is about 1.5.

It's interesting to note that a Koch snowflake is sometimes called a *Koch island*, because the snowflake's outline looks somewhat like the coastline of an island.

Geologists have found that the fractal dimension of geologic features such as mountains, coastlines, faults, valleys, and river courses are related to the conditions under which they were formed and the processes to which they have been subjected.

Some Applications of Fractal Dimension

The fractal dimension of a mountain's contour line will be higher if the mountain is made of easily eroded rock or if it is in an area of high rainfall.

Scientists have also found that the fractal dimension of the surfaces of small particles such as soot and pharmaceuticals is related to their physical characteristics.

For example, the tendency of soot to adhere to lung surfaces and the rate of absorption of ingested pharmaceuticals are both related to their fractal dimensions.