8

Geometry

8.1 Perimeter and Area

Objectives

- Find the perimeter of basic polygons, including squares, rectangles, and triangles
- Apply the Pythagorean Theorem to a right triangle
- Find the area of basic polygons, including squares, rectangles, triangles, parallelograms, and trapezoids

Objectives

- Apply Heron’s Formula to find the area of a triangle
- Use π in the calculation of circumference and area of a circle
- Understand and apply the units that are commonly used in astronomical measurement, including astronomical units, light-years, and parsecs

Polygons

Polygons

Two-dimensional figures can be classified by the number of sides they have.

A **polygon** is a many-sided figure.

A **pentagon** is a five-sided figure, a hexagon is a six-sided figure, and an octagon is an eight-sided figure.

Polygons

Two-dimensional figures can be classified by the number of sides they have.

A **polygon** is a many-sided figure.

A **pentagon** is a five-sided figure, a hexagon is a six-sided figure, and an octagon is an eight-sided figure.

However, the names of polygons do not necessarily end with -*gon*!

Polygons

Two-dimensional figures can be classified by the number of sides they have.

A **polygon** is a many-sided figure.

A **pentagon** is a five-sided figure, a hexagon is a six-sided figure, and an octagon is an eight-sided figure.

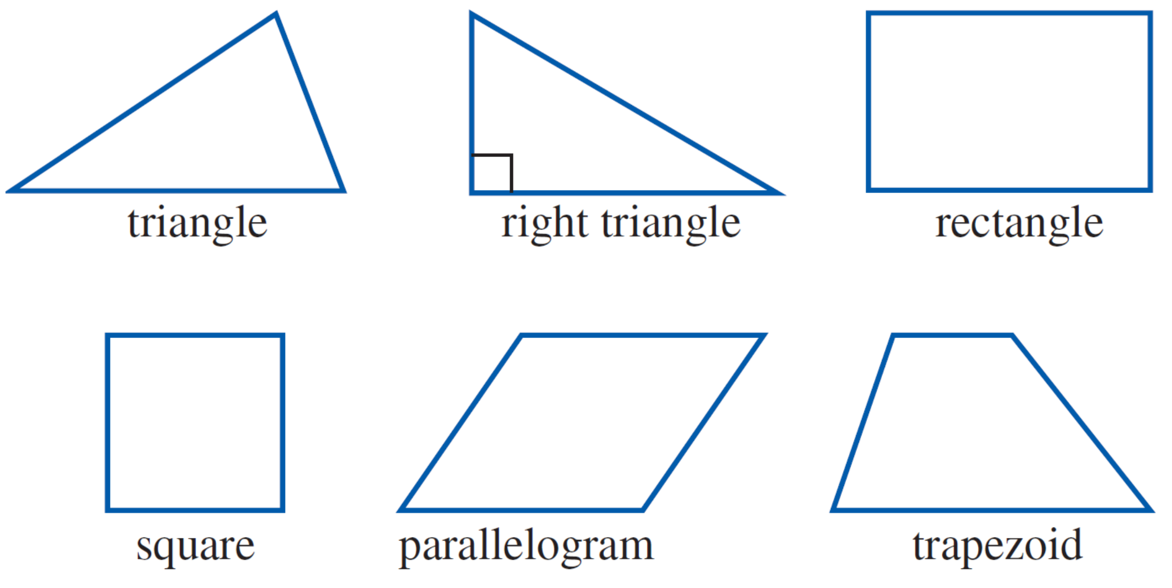
However, the names of polygons do not necessarily end with -*gon*!

A ***triangle*** is a three-sided figure.

A ***quadrilateral*** is a four-sided figure.

Polygons

Some of the polygons we will examine are shown in Figure 8.1. The symbol ⊾ represents an angle of 90° (90 degrees = a square corner = *right angle*).

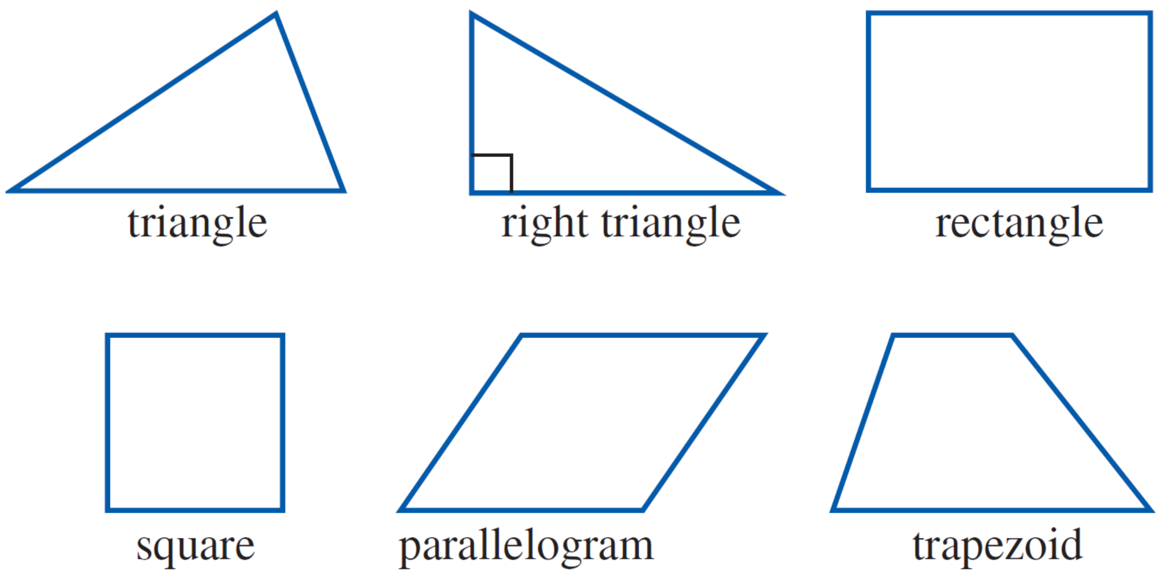


Common polygons.

Figure 8.1

Polygons

Some of the polygons we will examine are shown in Figure 8.1. The symbol ⊾ represents an angle of 90° (90 degrees = a square corner = *right angle*).

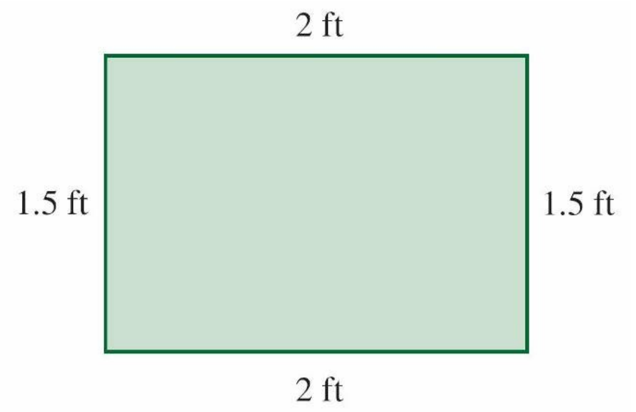


Which figures are quadrilaterals?

Polygons

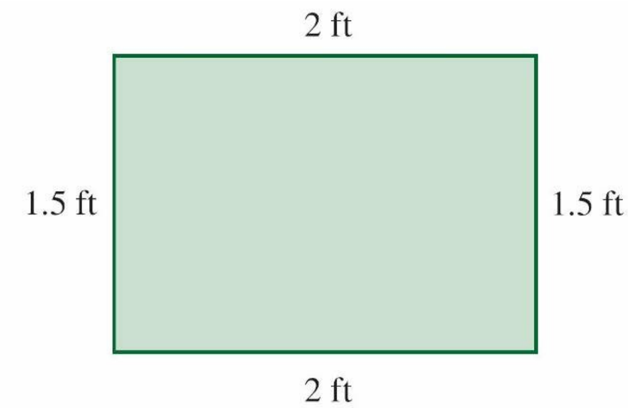
The **perimeter** of (or *distance around*) a two-dimensional figure is the sum of the lengths of its sides.

Perimeter = distance around



Polygons

The **perimeter** of (or *distance around*) a two-dimensional figure is the sum of the lengths of its sides.



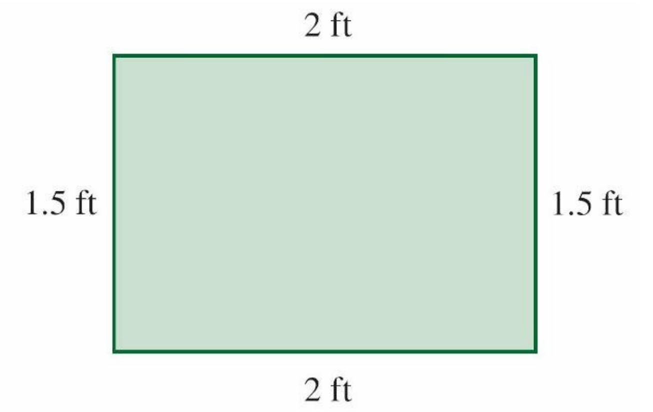
Perimeter = distance around

= 1.5 ft + 2 ft + 1.5 ft + 2 ft

= 7 ft

Polygons

The **perimeter** of (or *distance around*) a two-dimensional figure is the sum of the lengths of its sides.



Perimeter = distance around

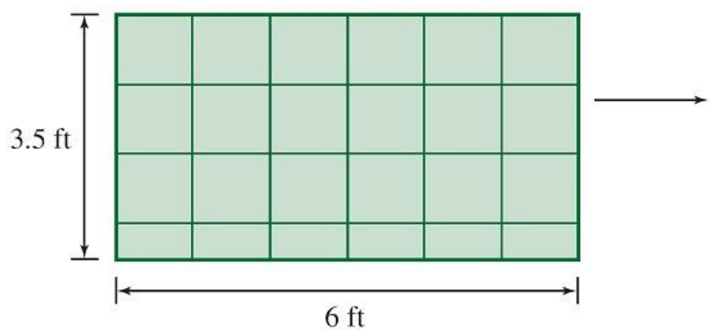
= 1.5 ft + 2 ft + 1.5 ft + 2 ft

= 7 ft

Important: same units of measurement should be used in the sum!

Polygons

The **area** of a two-dimensional figure is the number of square units (square inches, square miles, etc.) it takes to fill the interior of that figure.

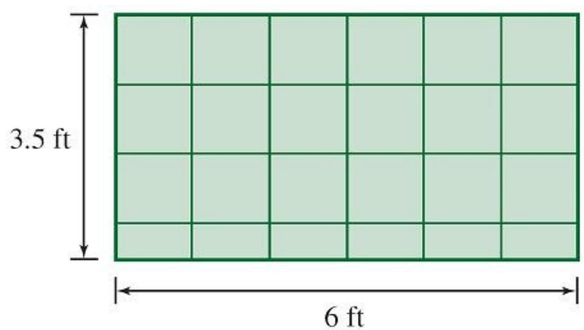


area = base × height

= length × width

Polygons

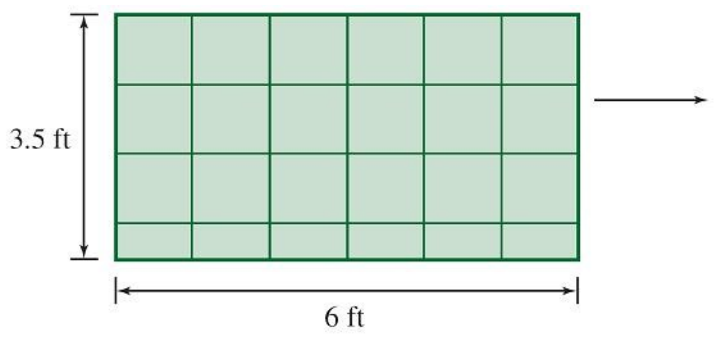
The **area** of a two-dimensional figure is the number of square units (square inches, square miles, etc.) it takes to fill the interior of that figure.



$$\begin{aligned} \text{area} &= \text{base} \times \text{height} \\ &= \text{length} \times \text{width} \\ &= (6 \text{ ft})(3.5 \text{ ft}) \\ &= 21 \text{ square ft} \\ &= 21 \text{ ft}^2 \end{aligned}$$

Polygons

The **area** of a two-dimensional figure is the number of square units (square inches, square miles, etc.) it takes to fill the interior of that figure.



$$\begin{aligned} \text{area} &= \text{base} \times \text{height} \\ &= \text{length} \times \text{width} \\ &= (6 \text{ ft})(3.5 \text{ ft}) \\ &= 21 \text{ square ft} \\ &= 21 \text{ ft}^2 \end{aligned}$$

Important: same units of measurement should be used in the product!

Polygons

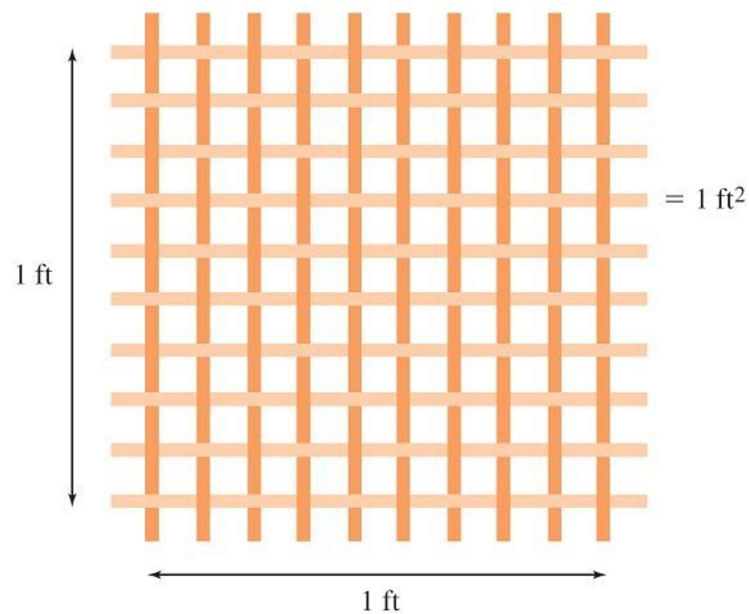
It has been suggested that the concepts of square units and area have their origins in the weaving of fabric.

Single strands of yarn have a linear, or one dimensional, measurement, such as inches or feet.

However, when an equal number of “horizontal” and “vertical” strands are woven together on a loom, a square figure is formed.

Polygons

Therefore, a natural way to measure the amount of cloth created from the strands is to employ units consisting of squares (square feet). (See Figure 8.4.)



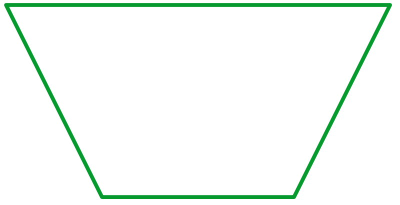
Woven cloth.

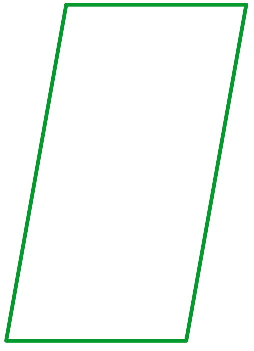
Figure 8.4

Polygons

On the basis of these quadrilaterals, we can also find the areas of triangles, trapezoids, and parallelograms.

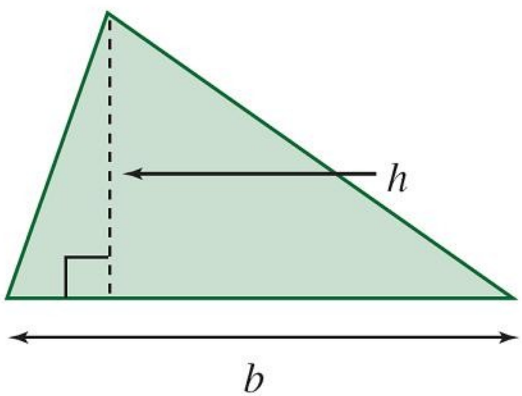
A **trapezoid** is a quadrilateral with one pair of parallel sides; a **parallelogram** is a quadrilateral with two pairs of parallel sides.





Polygons

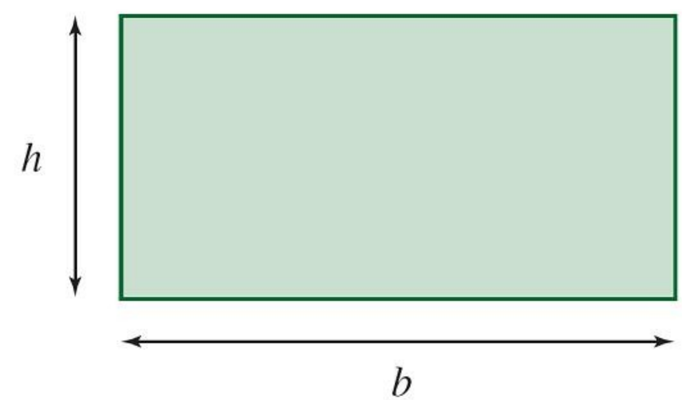
The area of a **triangle** can be found by the formula given below, with *A* = area, *b* = base, and *h* = height.



(a) **triangle** $A = \frac{1}{2}bh.$

Polygons

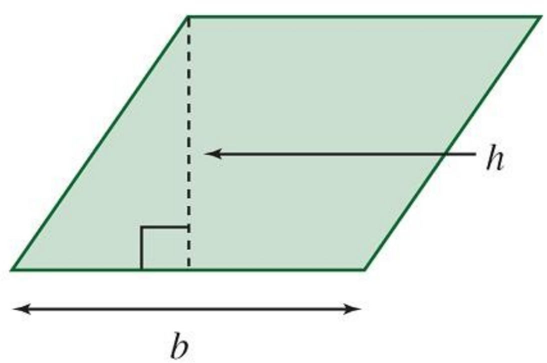
The area of a **rectangle** can be found by the formula given below, with A = area, b = base = length = l, and h = height = width = w.



(b) **rectangle** $A = bh = lw$

Polygons

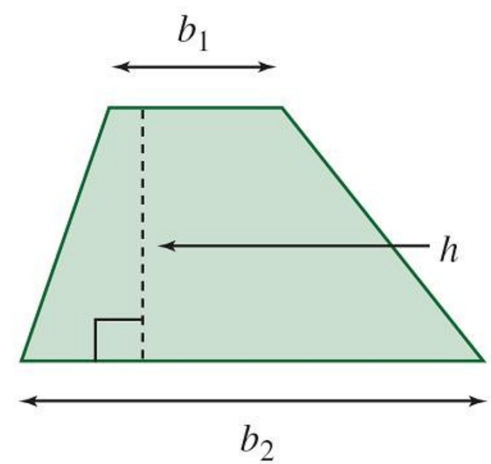
The area of a **parallelogram** can be found by the formula given below, with A = area, b = base, and h = height.



(c) **parallelogram** $A = bh.$

Polygons

The area of a **trapezoid** can be found by the formula given below, with A = area, h = height, and b_1 and b_2 = bases.



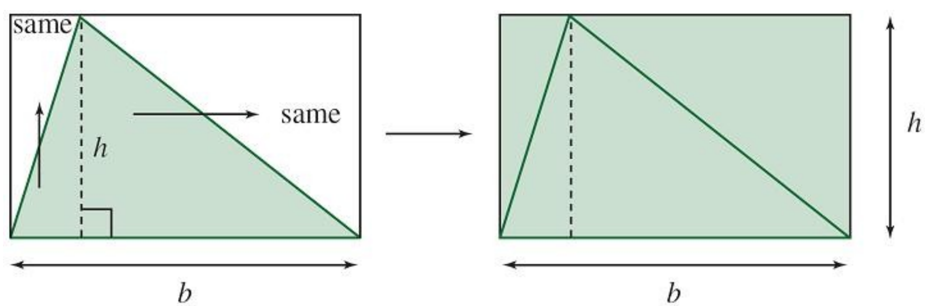
(d) **trapezoid** $A = \frac{1}{2}(b_1 + b_2)h.$

Polygons

<u>How the triangle area formula was derived</u>:

a triangle can be divided into two smaller triangles.

Copies of these smaller triangles are then "added on" to the original triangle to form a rectangle of area *b* • *h*



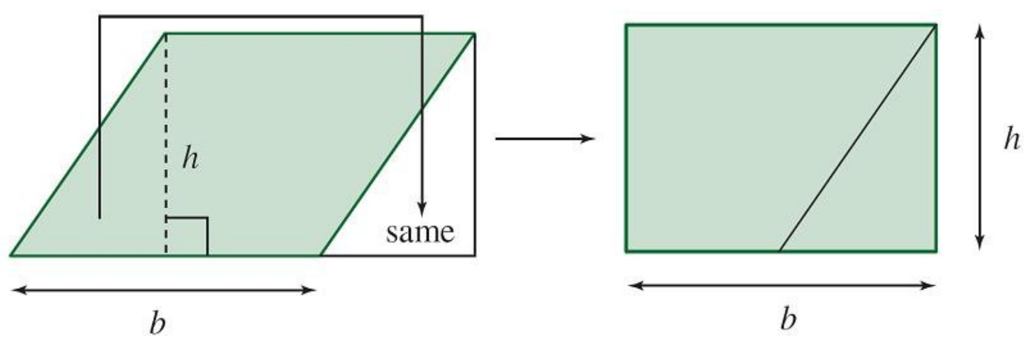
$$A_{\text{triangle}} = \frac{1}{2} A_{\text{rectangle}} = \frac{1}{2} bh.$$

Hence the area of the original triangle is <u>half that of the rectangle</u>, we have the desired result.

Polygons

<u>How the parallelogram area formula was derived</u>:

a parallelogram can be rearranged to form a rectangle of area *b* • *h*.



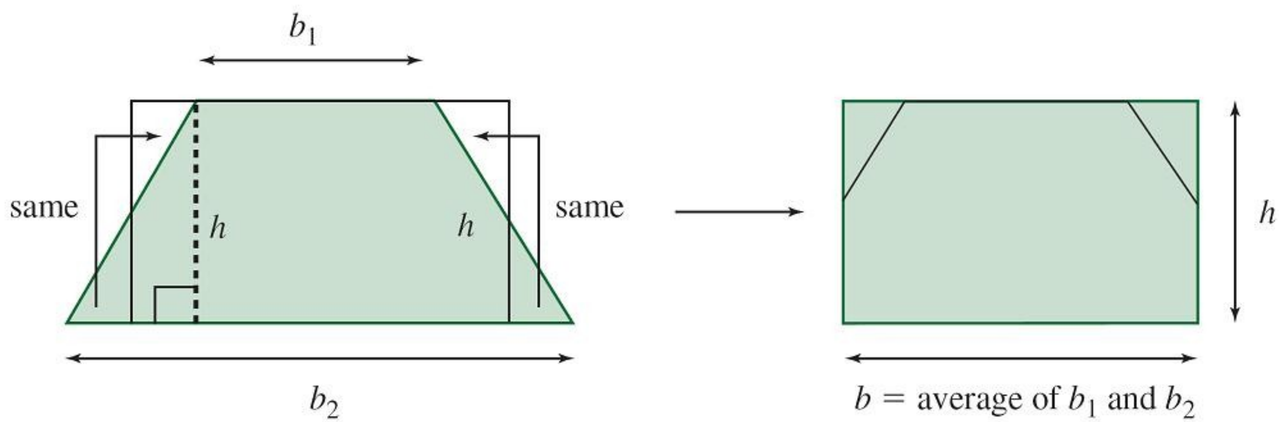
$$A_{\text{parallelogram}} = A_{\text{rectangle}} = bh.$$

That is, the area of the parallelogram is the same as the area of the rectangle, and we have the desired result.

Polygons

<u>How the trapezoid area formula was derived</u>:

The two triangular "tips" of the trapezoid can be cut off and rearranged to form a rectangle.



$$A_{\text{trapezoid}} = A_{\text{rectangle}} = \left(\frac{b_1 + b_2}{2}\right)(h).$$

The base of the rectangle equals the average of the two bases of the trapezoid; that is, $b_{\text{rectangle}} = (b_1 + b_2)/2$.

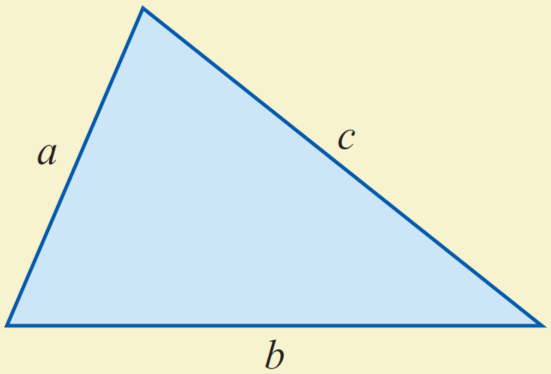
Heron’s Formula for the Area of a Triangle

Heron's Formula for the Area of a Triangle

Heron's Formula for the Area of a Triangle

$$s = \frac{1}{2}(a + b + c)$$

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$



Example 1 – *Finding Areas of Rectangles and Triangles*

Assuming the same growing conditions, which of the fields shown in Figures 8.9 and 8.10 would produce more grain?

a.

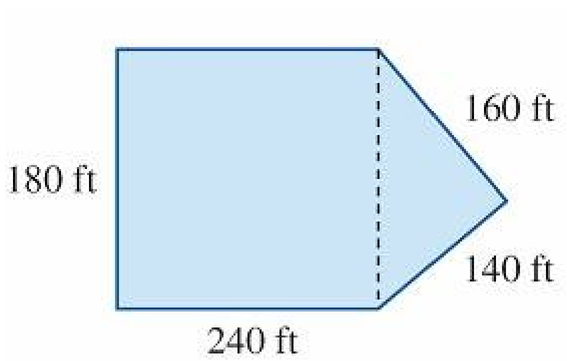


Figure 8.9

b.

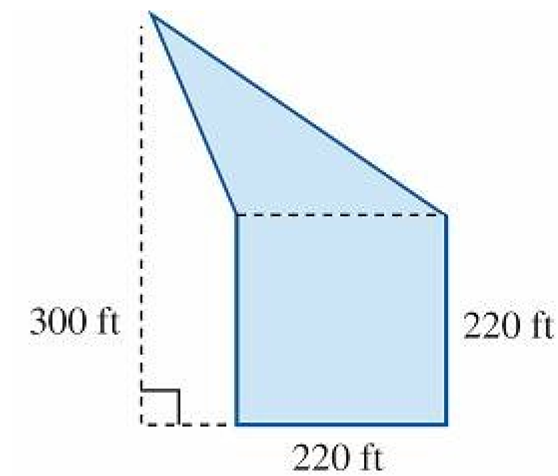


Figure 8.10

Example 1 – *Solution*

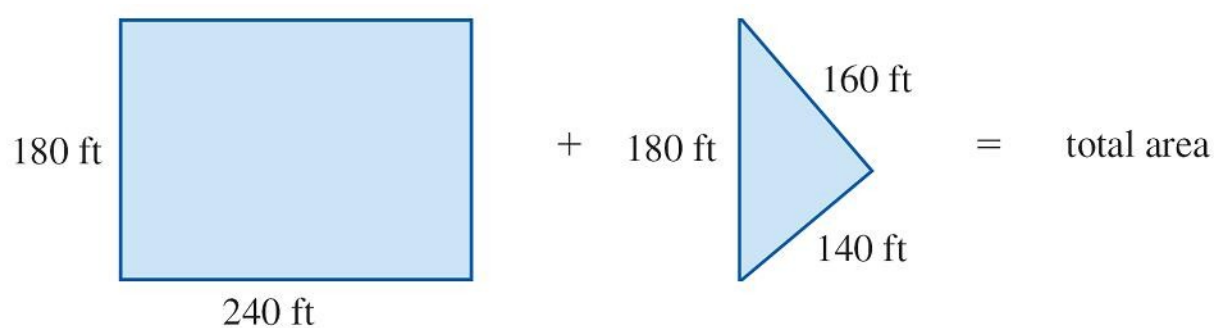
The field with the larger area would produce more grain!

Example 1 – *Solution*

The field with the larger area would produce more grain!

a. Finding the Area of the Field

The field consists of a rectangle and a triangle, as shown

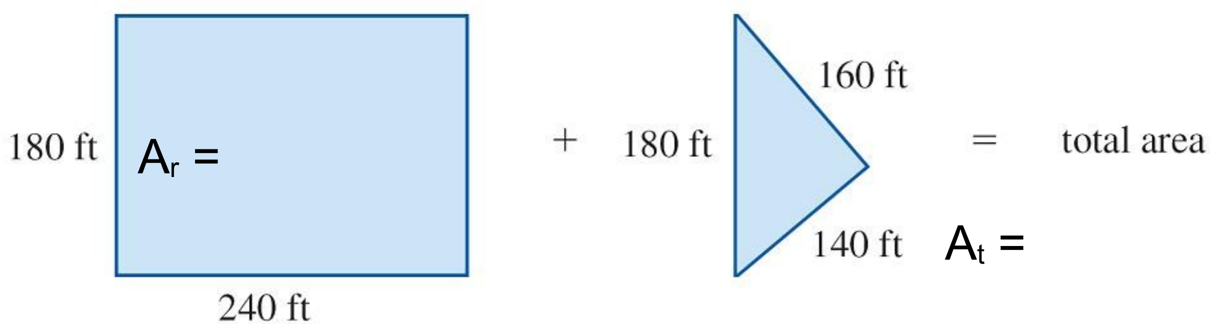


Example 1 – *Solution*

The field with the larger area would produce more grain!

a. Finding the Area of the Field

The field consists of a rectangle and a triangle, as shown

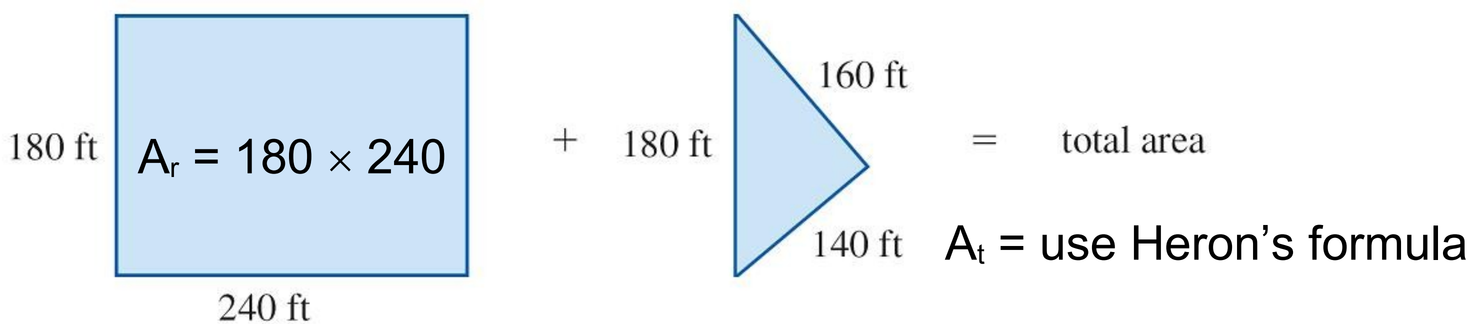


Example 1 – *Solution*

The field with the larger area would produce more grain!

a. Finding the Area of the Field

The field consists of a rectangle and a triangle, as shown

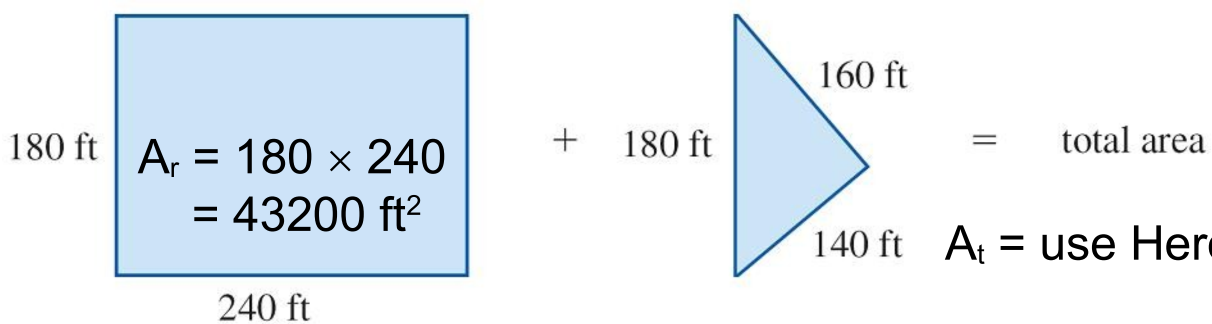


Example 1 – *Solution*

The field with the larger area would produce more grain!

a. Finding the Area of the Field

The field consists of a rectangle and a triangle, as shown



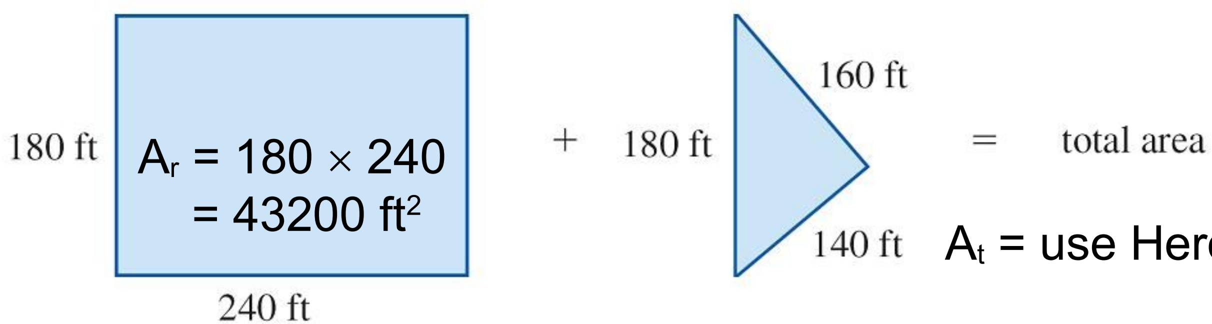
A_t = use Heron’s formula

Example 1 – *Solution*

The field with the larger area would produce more grain!

a. Finding the Area of the Field

The field consists of a rectangle and a triangle, as shown



$A_r = 180 \times 240$
$= 43200 \text{ ft}^2$

A_t = use Heron's formula

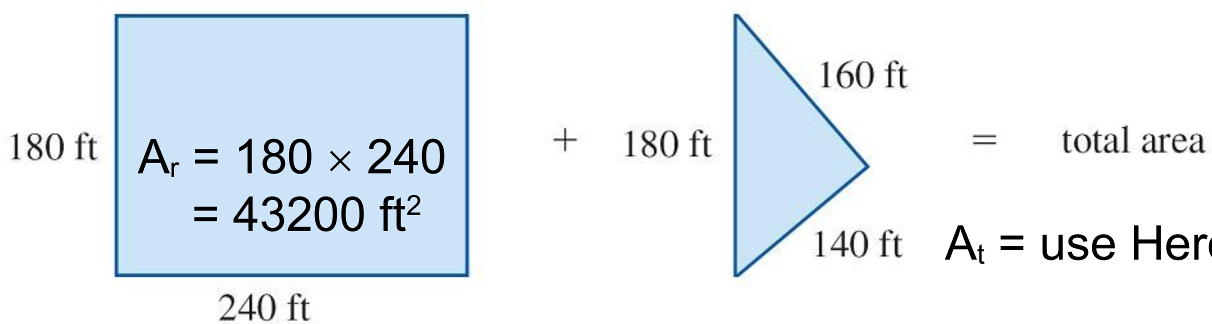
$$s = \frac{1}{2}(a + b + c)$$

Example 1 – *Solution*

The field with the larger area would produce more grain!

a. Finding the Area of the Field

The field consists of a rectangle and a triangle, as shown



$A_r = 180 \times 240$

$= 43200 \text{ ft}^2$

A_t = use Heron's formula

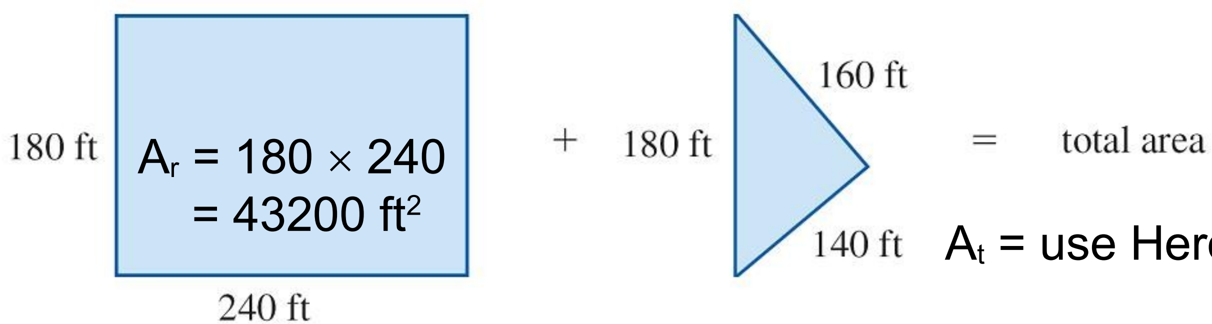
$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(180+160+140) = 240\,ft$$

Example 1 – *Solution*

The field with the larger area would produce more grain!

a. Finding the Area of the Field

The field consists of a rectangle and a triangle, as shown



A_t = use Heron's formula

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(180+160+140) = 240\, ft$$

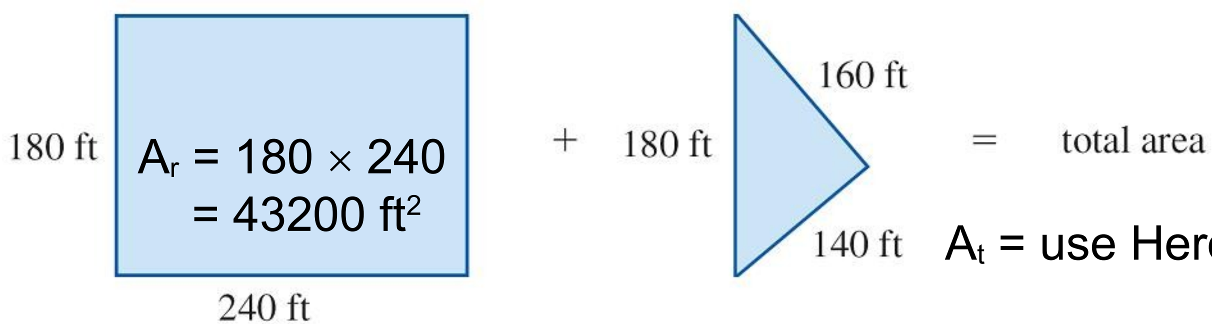
$$A_t = \sqrt{s(s-a)(s-b)(s-c)}$$

Example 1 – *Solution*

The field with the larger area would produce more grain!

a. Finding the Area of the Field

The field consists of a rectangle and a triangle, as shown



$A_r = 180 \times 240$
$= 43200\ ft^2$

A_t = use Heron's formula

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(180+160+140) = 240\, ft$$

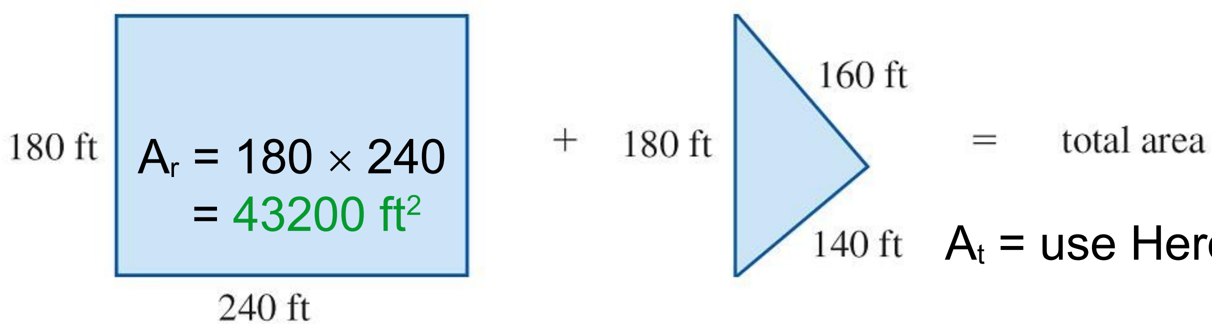
$$A_t = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{240(240-180)(240-160)(240-140)} \approx 10733\, ft^2$$

Example 1 – *Solution*

The field with the larger area would produce more grain!

a. Finding the Area of the Field

The field consists of a rectangle and a triangle, as shown



$A_r = 180 \times 240$

$= 43200 \text{ ft}^2$

A_t = use Heron’s formula

$$s=\frac{1}{2}(a+b+c)=\frac{1}{2}(180+160+140)=240\,ft$$

$$A_t=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{240(240-180)(240-160)(240-140)}\approx 10733\,ft^2$$

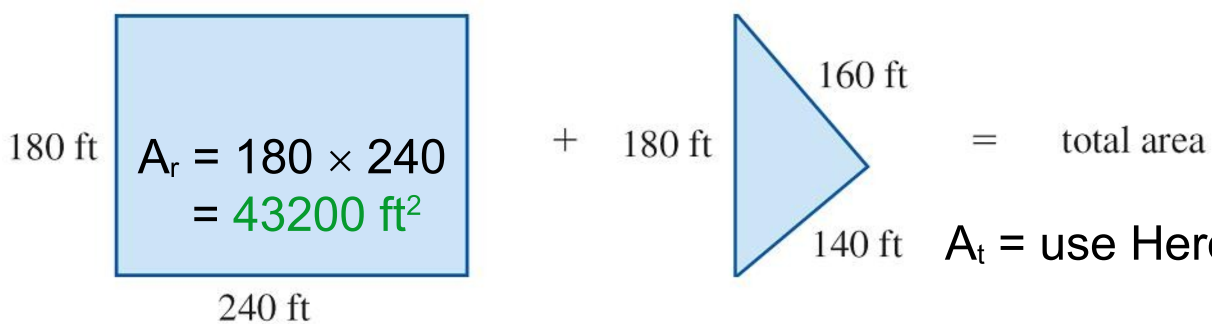
Example 1 – *Solution*

The field with the larger area would produce more grain!

a. Finding the Area of the Field

The field consists of a rectangle and a triangle, as shown

Area of the field = 43200 + 10733 = 53933 ft^2



$A_r = 180 \times 240$
$= 43200 \text{ ft}^2$

A_t = use Heron's formula

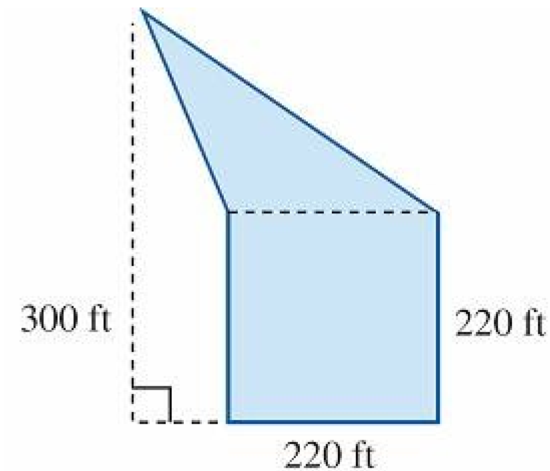
$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(180+160+140) = 240\,ft$$

$$A_t = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{240(240-180)(240-160)(240-140)} \approx 10733\,ft^2$$

Example 1 – *Solution*

b. Finding the Area of the Field

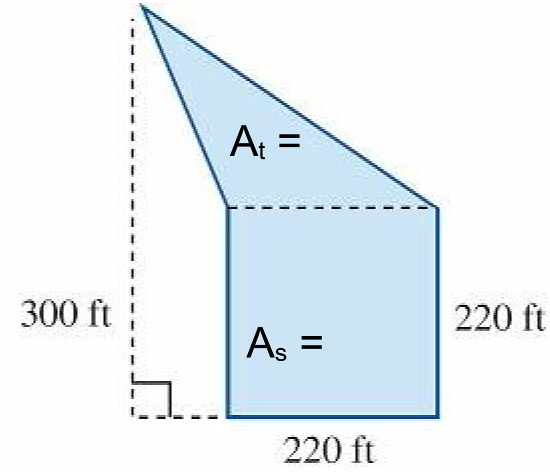
The field in part (b) is composed of a square and a triangle.



Example 1 – *Solution*

b. Finding the Area of the Field

The field in part (b) is composed of a square and a triangle.

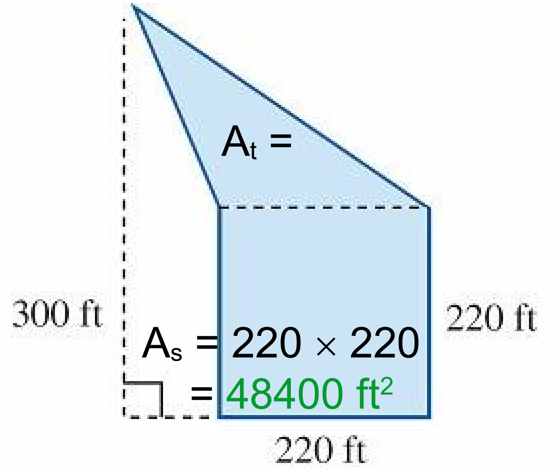


Example 1 – *Solution*

cont'd

b. Finding the Area of the Field

The field in part (b) is composed of a square and a triangle.

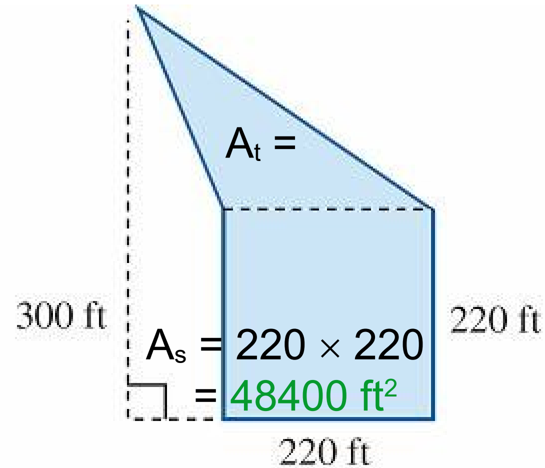


Example 1 – *Solution*

cont'd

b. Finding the Area of the Field

The field in part (b) is composed of a square and a triangle.



The lengths of two of the sides of the triangle are not given, so the semi-perimeter s cannot be used.

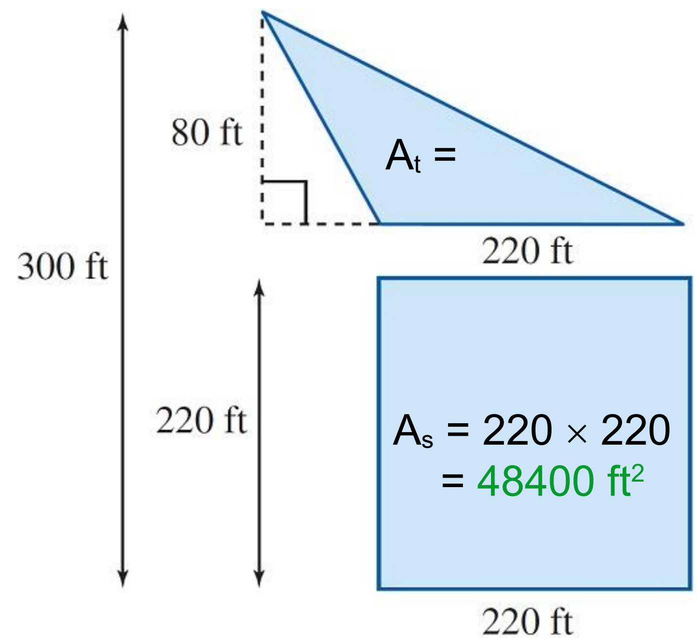
However, we do know that the square at the bottom is 220 ft high, and the height up to the tip of the figure is 300 ft. Hence, the triangle's height = 300 ft – 220 ft = 80 ft

Example 1 – *Solution*

cont'd

b. Finding the Area of the Field

The field in part (b) is composed of a square and a triangle.



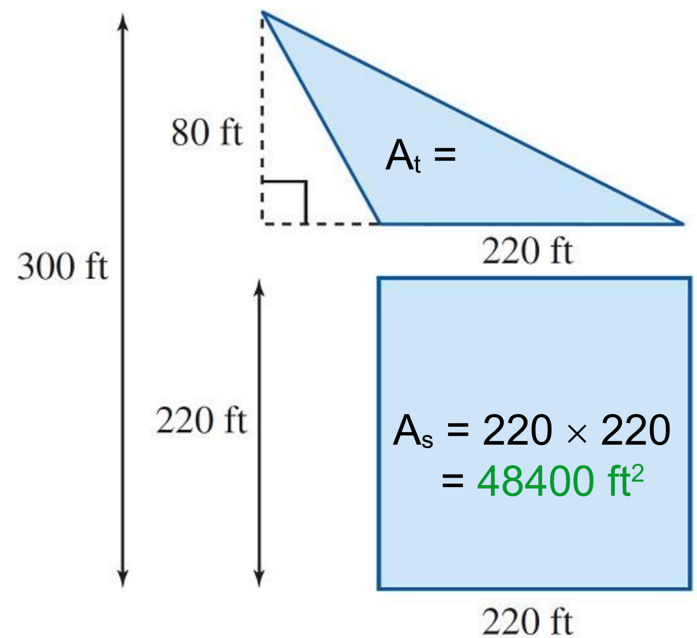
Example 1 – *Solution*

cont'd

b. Finding the Area of the Field

The field in part (b) is composed of a square and a triangle.

$$A_t = \frac{1}{2}hb$$



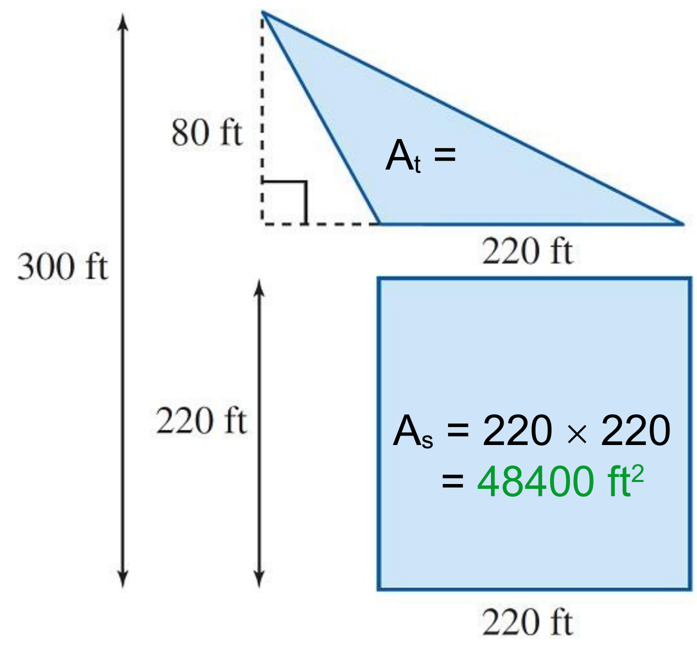
Example 1 – *Solution*

cont'd

b. Finding the Area of the Field

The field in part (b) is composed of a square and a triangle.

$$A_t = \frac{1}{2}hb = \frac{1}{2}80 \times 220 = 8800\,ft$$



Example 1 – *Solution*

cont'd

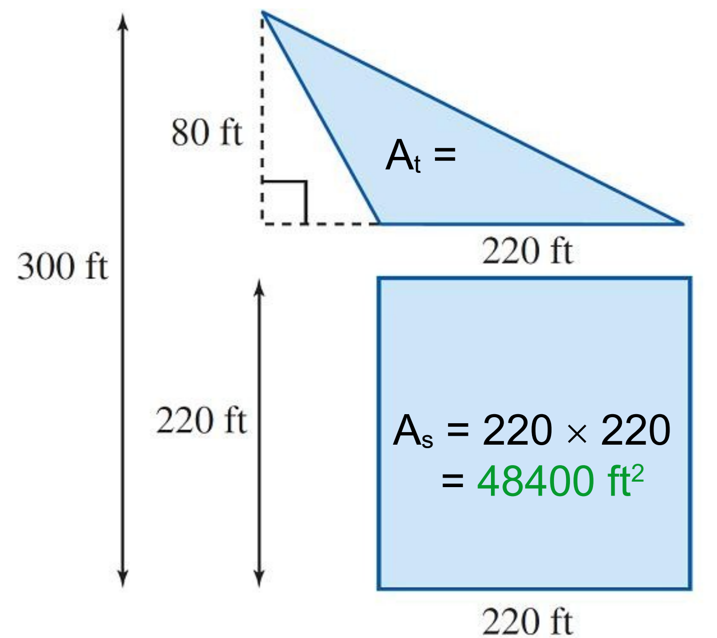
b. Finding the Area of the Field

The field in part (b) is composed of a square and a triangle.

$$A_t = \frac{1}{2}hb = \frac{1}{2}80 \times 220 = 8800\,ft$$

Area of the field = 48400 + 8800

= 57200 ft^2



Example 1 – *Solution*

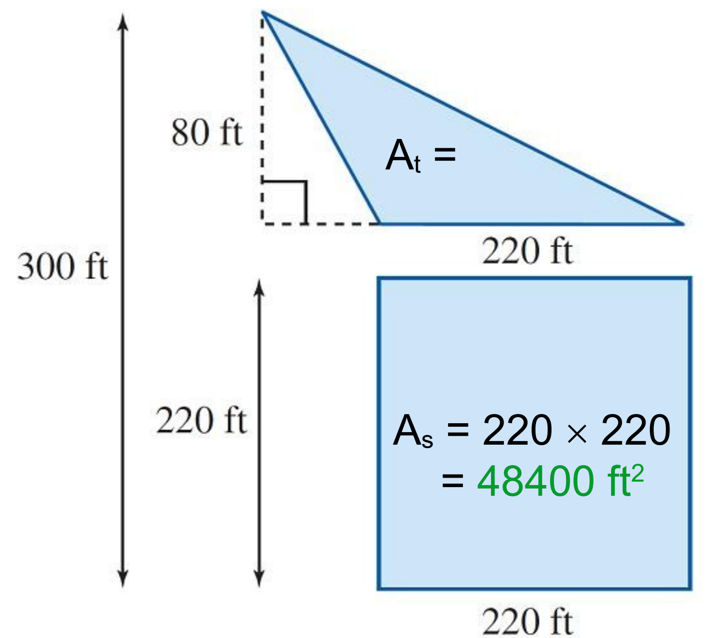
b. Finding the Area of the Field

The field in part (b) is composed of a square and a triangle.

$$A_t = \frac{1}{2}hb = \frac{1}{2}80 \times 220 = 8800\,ft$$

Area of the field = 48400 + 8800
= 57200 ft^2

Area of the field in a) = 53933 ft^2



Example 1 – *Solution*

cont'd

b. Finding the Area of the Field

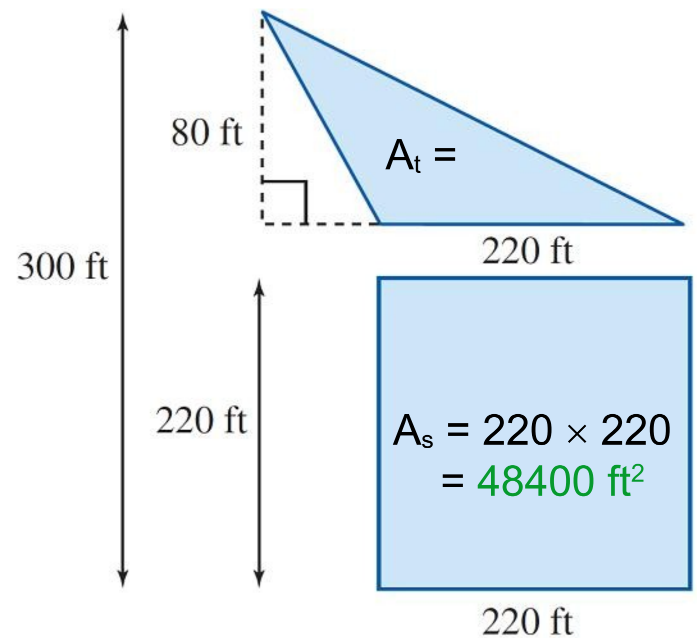
The field in part (b) is composed of a square and a triangle.

$$A_t = \frac{1}{2}hb = \frac{1}{2}80 \times 220 = 8800\,ft$$

Area of the field = 48400 + 8800

= 57200 ft^2

Area of the field in a) = 53933 ft^2



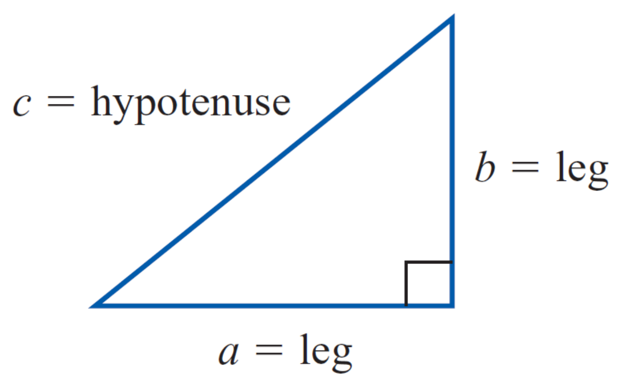
<u>Answer</u>: Assuming the same growing conditions, the field in part (b) would produce more grain, because it has a larger area than the field in part (a).

Right Triangles

Right Triangles

If one of the angles of a triangle is a right angle (a square corner, or 90°), the triangle is called a ***right triangle***.

The side opposite the right angle is called the **hypotenuse** and is labeled *c*, while the remaining two sides are called the **legs** and are labeled *a* and *b*.



Right triangle.

Right Triangles

A special relationship exists between the hypotenuse and the legs of a right triangle. Over 2,000 years ago, early geometers observed that if the longest side of a right triangle (the hypotenuse) was squared, the number obtained was always the same as the sum of the squares of the two other sides (the legs).

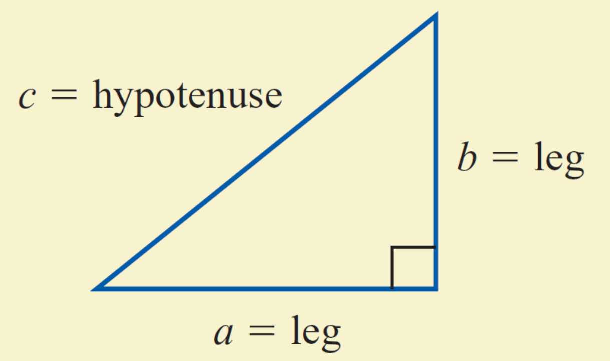
This observation was proved to be true for *all* right triangles. It is referred to as the **Pythagorean theorem**, in honor of the ancient Greek mathematician Pythagoras of Samos (circa 580 b.c.), although the result was known to earlier peoples.

Right Triangles

Pythagorean Theorem

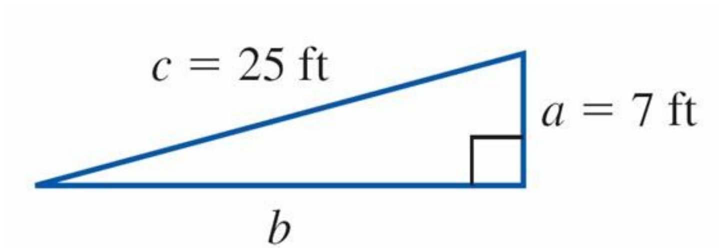
For any right triangle, the square of the hypotenuse equals the sum of the squares of the legs.

$$c^2 = a^2 + b^2$$



Example 3 – *Applying the Pythagorean Theorem and Finding the Area of a Right Angle Triangle*

The length of the hypotenuse of a right triangle is 25 feet, and the length of one of the legs is 7 feet.
Find the area of the triangle.



A right triangle.

Example 3 – *Solution*

To find the area of a triangle, we must know either a base and its corresponding (perpendicular) height or all three sides.

In this case, we must find the missing side of the given triangle. Because we have a right triangle, we can apply the Pythagorean Theorem to find the missing leg, b:

$$a^2 + b^2 = c^2$$

$$b^2 = c^2 - a^2$$

$$b^2 = (25)^2 - (7)^2$$

Example 3 – *Solution*

cont'd

$$b^2 = 625 - 49$$

$$= 576$$

$$b = \sqrt{576}$$

$$= 24 \text{ ft}$$

Now we find the area of the triangle:

$$A = \frac{1}{2}bh$$

$$= \frac{1}{2}(24 \text{ ft})(7 \text{ ft})$$

$$= 84 \text{ ft}^2$$

The area of the triangle is 84 square feet.

Circles

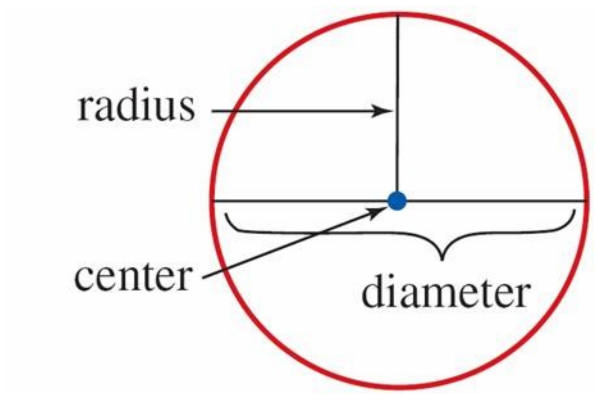
Circles

Many people consider the circle a perfect geometric figure; it has no beginning or end, it has total symmetry, and it motivated the invention of the famous irrational (nonfraction) number π **(pi).** The early Greeks defined a **circle** as the set of all points in a plane equidistant from a fixed point.

The fixed point is called the **center** of the circle;
a line segment from the center to any point on the circle is called a **radius;**
and any line segment connecting two points on the circle and passing through the center is called a **diameter.**

Circles

(The length of a diameter is twice the length of a radius.)

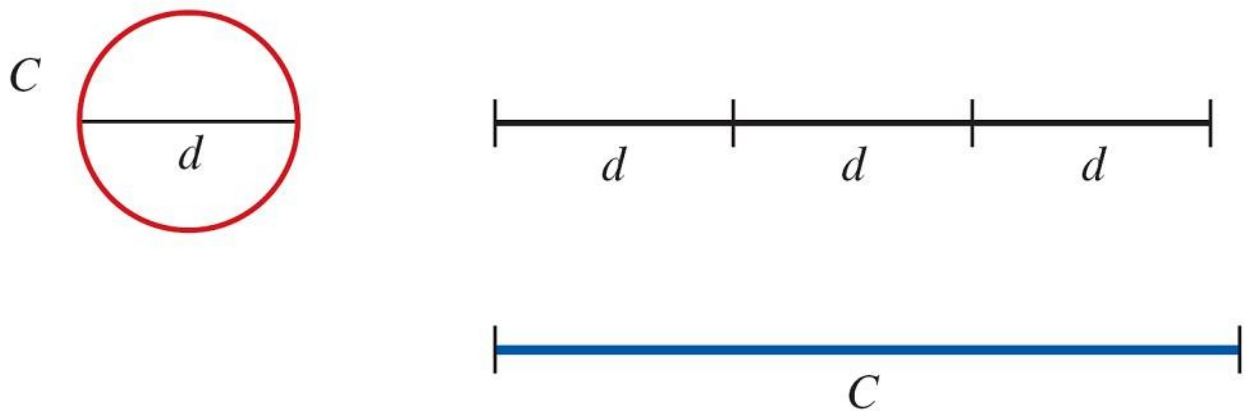


A circle.

Recall that the distance around a figure is called its *perimeter*; however, as a special case, the distance around a circle is called its **circumference.**

Circles

Thousands of years ago, geometers observed a curious relationship: If they measured the circumference C of *any* circle (probably by using a string or a rope), they found that it was always a little bit longer than 3 times the diameter d of the circle.



Circumference of a circle.

Circles

In other words, the ratio of circumference to diameter is constant: circumference/diameter = constant. This constant number, which is a little larger than 3, is represented by the Greek letter π (read "pi"; rhymes with *sly*). Hence,

$$\frac{\text{circumference}}{\text{diameter}} = \pi$$

or

$$\text{circumference} = \pi \cdot \text{diameter}$$

Circles

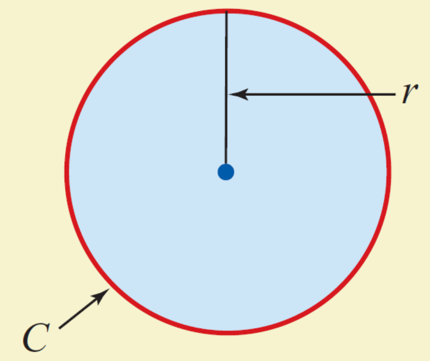
Circumference and Area of a Circle

The **circumference** C of a circle of radius r is

$$C = 2\pi r$$

The **area** A of a circle of radius r is

$$A = \pi r^2$$

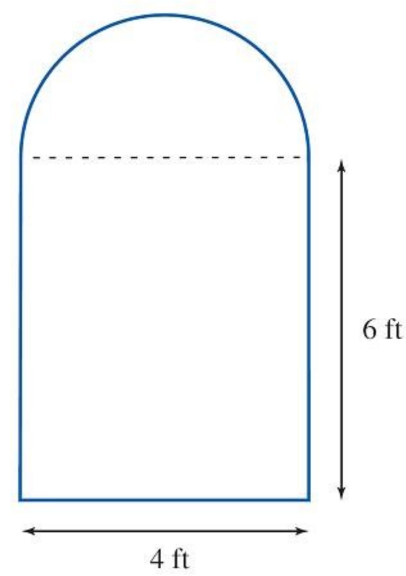


Example 4 – *Finding the Area and Perimeter of a Hybrid Shape*

A Norman window consists of a rectangle with a semicircle mounted on top. For the window shown in Figure 8.19, find the following.

a. the area

b. the perimeter



A Norman window.

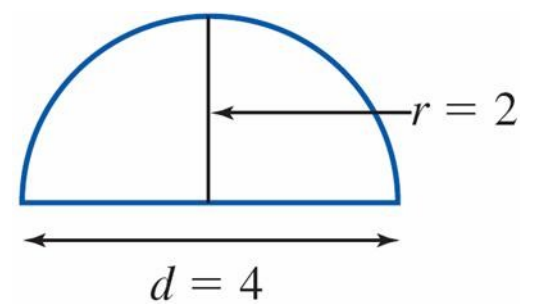
Figure 8.19

Example 4 – *Solution*

a. The area of the entire window is the sum of the areas of the rectangle and the semicircle. A semicircle is half a circle, so the semicircular region has area equal to one half the area of a complete circle.

The base of the rectangle is the same as the diameter of the semicircle.

Hence, the diameter is 4 feet, and the radius is 2 feet.



Half a circle.

Example 4 – *Solution*

cont'd

$$A_{\text{total}} = A_{\text{rectangle}} + A_{\text{semicircle}}$$

$$= bh + \frac{1}{2}(\pi r^2)$$

$$= (4\text{ ft})(6\text{ ft}) + \frac{1}{2}\pi(2\text{ ft})^2$$

$$= 24\text{ ft}^2 + 2\pi\text{ ft}^2$$

$$= (24 + 2\pi)\text{ ft}^2$$

$$= 30.28318531\ldots\text{ ft}^2$$

Thus, the area of the Norman window is approximately 30.3 square feet.

Example 4 – *Solution*

cont'd

b. The perimeter of the window consists of a semicircle (one-half the circumference of a circle), one horizontal line segment, and two vertical line segments.

$$P = \frac{1}{2}C + b + 2h$$

$$= \frac{1}{2}(\pi \cdot 4 \text{ ft}) + 4 \text{ ft} + 2(6 \text{ ft})$$

$$= 2\pi \text{ ft} + 16 \text{ ft}$$

$$= 22.28318531 \ldots \text{ ft}$$

Thus, the perimeter of the Norman window is approximately 22.3 feet.