5

Finance

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5.3

Annuities

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Objectives

- Understand what an annuity is
- Use the Annuity Formulas
- Determine how to use an annuity to save for retirement

Annuities

An **annuity** is simply a sequence of equal, regular payments into an account in which each payment receives compound interest.

Because most annuities involve relatively small periodic payments, they are affordable for the average person.

Over longer periods of time, the payments themselves start to amount to a significant sum, but it is really the power of compound interest that makes annuities so amazing.

Annuities

If you pay \$50 a month into an annuity for the next forty years, then you will have about \$300,000 in your account, even though your total payment is

$$\frac{\$50}{\cancel{\text{month}}} \cdot \frac{12\ \cancel{\text{months}}}{\cancel{\text{year}}} \cdot 40\ \cancel{\text{years}} = \$24{,}000$$

The interest portion is

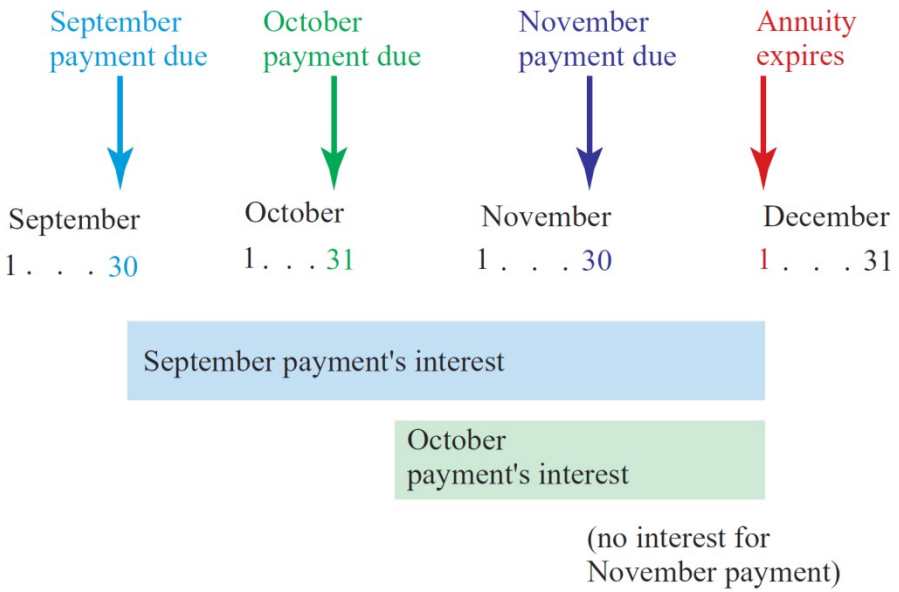
$$\$300{,}000 - \$24{,}000 = \$276{,}000$$

Annuities as Compound Interest, Repeated

Example 1 – *Understanding Annuities*

On September 12, Patty Leitner started an annuity. She arranged to have $200 taken out of each of her end-of-the month paychecks. The money would earn $8\frac{3}{4}\%$ interest compounded monthly.

Find the future value of the account on November 30 by applying the Compound Interest Formula to the September payment, the October payment, and the November payment.



Patty's payment schedule and interest earning periods.

Figure 5.11

Example 1 – *Solution*

A timeline for Patty's payments is shown in Figure 5.11.

We are given $P = 200$ and $r = \frac{1}{12}^{th} \text{ of } 8\frac{3}{4}\% = \frac{0.0875}{12}$

To find the future value of the first payment (made on September 30), use $n = 2$, because the payment will receive interest during October and November.

$FV = P(1 + i)^n$ the Compound Interest Formula

$= 200\left(1 + \frac{0.0875}{12}\right)^2$ substituting

$= 202.9273 \approx \$202.93$ rounding

Example 1 – *Solution*

cont'd

To calculate the future value of the second payment (made on October 31), use $n = 1$. This payment will receive interest during November.

$FV = P(1 + i)^n$ the Compound Interest Formula

$= 200\left(1 + \frac{0.0875}{12}\right)^1$ substituting

$= 201.45833$

$\approx \$201.46$ rounding

Example 1 – *Solution*

cont'd

We are to find the future value of the account on November 30. The third payment is made on November 30, so it earns no interest (until after November 30, which is beyond the scope of the example). Therefore,

$$FV = \$200$$

The future value of Patty's annuity is:

$$FV = \$200 + \$201.46 + \$203.93 = \$604.39$$

Patty made three payments of $200 each, so her total contribution was $600. She earned $4.39 interest on these payments.

The Annuity Formula

The Annuity Formula

The procedure followed in Examples 1 reflects what actually happens with annuities, and it works fine for a small number of payments.

However, most annuities are long-term, and the procedure would be impractical if we were computing the future value after forty years.

Because of this, long-term annuities are calculated with their own formula.

To get that formula, we can generalize on the work we did in Example 1. However, we will skip it and go directly to ...

The Annuity Formula

For an annuity with payment *pymt*, a periodic rate *i*, and a term of *n* payments, its future value is:

$$FV = \frac{pymt[(1 + i)^n - 1]}{i}$$

Two Types of Annuities

Two Types of Annuities

There are two types of annuities:

- ***ordinary annuity***, with payments due at the end of the month (at the end of each period)

- ***annuity due***, with payments due at the beginning of each period.

In ***annuity due***, each payment is due one month earlier, so each payment receives interest for one more month. This means that the future value of an annuity due is the *future value of the ordinary annuity, plus one more period's interest*.

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$$FV\left(ordinary\right)=\frac{pymt\left[\left(1+i\right)^{n}-1\right]}{i}$$

- ***annuity due***, with payments due at the beginning of each period.

$$FV\left(due\right)=FV\left(ordinary\right)+one\ period\ 's\ interest$$

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$$FV\,(ordinary) = \frac{pymt\left[(1+i)^n - 1\right]}{i}$$

- ***annuity due***, with payments due at the beginning of each period.

$$FV\,(due) = FV\,(ordinary) + one\ period's\ interest$$

$$= FV(\text{ordinary}) + FV(\text{ordinary}) \cdot i$$

to get one period's interest, multiply by *i*

$$= FV(\text{ordinary})(1 + i)$$

factoring

Two Types of Annuities

Annuity

An **annuity** is a sequence of regular equal payments into an account where each payment receives compound interest.

The future value of an **ordinary annuity** with payment size *pymt*, periodic rate *i*, and a term of *n* payments is:

$$FV(\text{ord}) = pymt\frac{[(1+i)^n - 1]}{i}$$

The future value of an **annuity due** is:

$$FV(\text{due}) = FV(\text{ordinary})(1+i)$$

$$= pymt\frac{[(1+i)^n - 1]}{i}(1+i)$$

Tax-Deferred Annuities

Tax-Deferred Annuities

A **tax-deferred annuity (TDA)** is an annuity that is set up to save for retirement. Money is automatically deducted from the participant's paychecks until retirement, and the federal (and perhaps state) tax deduction is computed *after* the annuity payment has been deducted, resulting in significant tax savings.

In some cases, the employer also makes a regular contribution to the annuity.

The next Example involves a long-term annuity. Usually, the interest rate of a long-term annuity varies somewhat from year to year. In this case, calculations must be viewed as predictions, not guarantees.

Example 2 – *Using an Annuity to Save for Retirement*

Tom and Betty decided that they should start saving for retirement, so they set up a tax-deferred annuity. They arranged to have $200 taken out of each of Tom's monthly checks, which will earn $8\frac{3}{4}\%$ interest.

Because of the tax-deferring effect of the TDA, Tom's take-home pay went down by only $115. Tom just had his thirtieth birthday, and his ordinary annuity will come to term when he is 65.

a. Find the future value of the annuity.

b. Find Tom's contribution and the interest portion.

Example 2(a) – *Solution*

This is an ordinary annuity with *pymt* = 200,

$$i = \tfrac{1}{12}\text{th of } 8\tfrac{3}{4}\% = \frac{0.0875}{12}$$

n = 35 years

= 35 years • 12 months/year

= 420 monthly payments.

$$FV(\text{ord}) = pymt\frac{(1+i)^n - 1}{i}$$

the Ordinary Annuity Formula

$$= 200\frac{(1+0.0875/12)^{420} - 1}{0.0875/12}$$

substituting

≈ $552,539.96

rounding

Example 2(b) – *Solution*

cont'd

The principal part of this $552,539.96 is Tom's contribution, and the rest is interest.

- *Tom's contribution* is 420 payments of $200 each

$$= 420 \cdot \$200$$

$$= \$84{,}000.$$

- The *interest portion* is then

$$\$552{,}539.96 - \$84{,}000 = \$468{,}539.96.$$

Sinking Funds

Sinking Funds

A **sinking fund** is an annuity in which the future value is a specific amount of money that will be used for a certain purpose, such as a child's education or the down payment on a home.

Example 3 – *Using an Annuity to Save a Specific Amount*

Tom and Betty have a new baby. They agreed that they would need $30,000 in eighteen years for the baby's college education. They decided to set up a sinking fund and have money deducted from each of Betty's biweekly paychecks. That money will earn $9\frac{1}{4}\%$ interest in Betty's ordinary annuity. Find their biweekly payment.

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Solution:

This is an ordinary annuity, with $i = \frac{1}{26}\text{th of } 9\frac{1}{4}\% = 0.0925/26$, and $n = 18$ years • 26 periods/year = 468 periods, and $FV = \$30{,}000$.

Example 3 – *Solution*

cont'd

$$FV(\text{ord}) = pymt\,\frac{(1+i)^n - 1}{i}$$

the Ordinary Annuity Formula

$$\$30{,}000 = pymt\,\frac{(1+0.0925/26)^{468} - 1}{0.0925/26}$$

substituting

To find *pymt*, we must divide 30,000 by the fraction on the right side of the equation.

Because the fraction is so complicated, it is best to first calculate the fraction and then multiply its reciprocal by 30,000.

Example 3 – *Solution*

cont'd

This gives *pymt* = 24.995038…. Betty would need to have only $25 taken out of each of her biweekly paychecks to save $30,000 in eighteen years.

Notice that she will not have exactly $30,000 saved, because she cannot have exactly $24.995048… deducted from each paycheck.

Present Value of an Annuity

Present Value of an Annuity

The **present value of an annuity** is the lump sum that can be deposited at the beginning of the annuity's term, at the same interest rate and with the same compounding period, that would yield the same amount as the annuity.

This value can help the saver to understand his or her options; it refers to an alternative way of saving the same amount of money in the same time.

It is called the *present value* because it refers to the single action that the saver can take *in the present* (i.e., at the beginning of the annuity's term) that would have the same effect as would the annuity.

Example 4 – *Finding the Present Value*

Find the present value of an ordinary annuity that has $200 monthly payments for 25 years, where the account receives $10\frac{1}{2}\%$ interest.

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Calculate the right side first! Then divide the result by

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P = 21182.363… P ≈ $21,182.36

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P ≈ $21,182.36 – the lump sum deposit to have as much money as after 25 years with monthly payment of $200.

Present Value of an Annuity

Present Value of Annuity Formula

$$FV(\text{lump sum}) = FV(\text{annuity})$$

$$P(1 + i)^n = pymt\frac{(1 + i)^n - 1}{i}$$

The present value is the lump sum P.