2

Sets and Counting

2.5 Infinite Sets

Objectives

- Determine whether two sets are equivalent
- Establish a one-to-one correspondence between the elements of two sets
- Determine the cardinality of various infinite sets.

One-to-One Correspondence

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Is there any relationship between the sets

A = {one, two, three}

and

B = {Buick, Chevrolet, Ford}?

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B = {Buick, Chevrolet, Ford}?

Although the sets contain different types of things, each contains the same number of things; they are the same size.

This relationship (being the same size) forms the basis of a *one-to-one correspondence*.

One-to-One Correspondence

A ***one-to-one correspondence*** between the sets *A* and *B* is a pairing up of the elements of *A* and *B* such that each element of *A* is paired up with exactly one element of *B,* and vice versa, with no element left out.

For instance, the elements of *A* and *B* might be paired up as follows:

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(Other correspondences, or match ups, are possible.)

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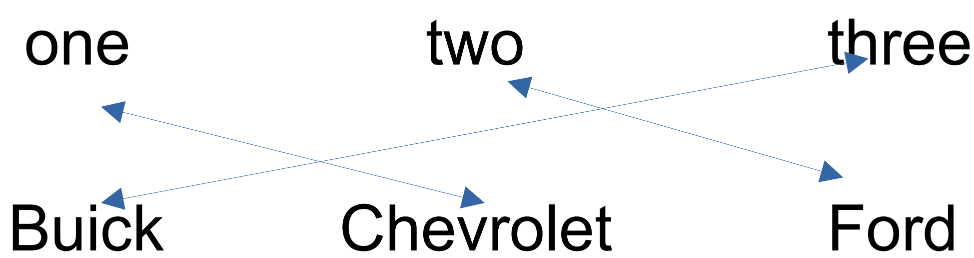

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If two sets have the same cardinal number, their elements can be put into a one-to-one correspondence.

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Whenever a one-to-one correspondence exists between the elements of two sets A and B, the sets are ***equivalent***

denotation: $A \sim B$

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Hence, *equivalent sets have the same number of elements*.

One-to-One Correspondence

Given two sets *A* and *B*, if any one of the following statements is true, then the other statements are also true:

1. There exists a one-to-one correspondence between the elements of *A* and *B.*

2. *A* and *B* are equivalent sets.

3. *A* and *B* have the same cardinal number; that is, $n(A) = n(B)$.

Example 1 – *Determining Whether Two Sets are Equivalent*

Determine whether the sets in each of the following pairs are equivalent. If they are equivalent, list a one-to-one correspondence between their elements.

a. A = {John, Paul, George, Ringo};

B = {Lennon, McCartney, Harrison, Starr}

b. $C = \{\alpha, \beta, x, \delta\}$;

$D = \{\text{I}, \text{O}, \Delta\}$

c. $A = \{1, 2, 3, \ldots, 48, 49, 50\}$;

$B = \{1, 3, 5, \ldots, 95, 97, 99\}$

Example 1 – *Solution*

a. If sets have the same cardinal number, they are equivalent. Now, $n(A) = 4$ and $n(B) = 4$; therefore, $A \sim B$.

Because A and B are equivalent, their elements can be put into a one-to-one correspondence. One such correspondence follows:

John	Paul	George	Ringo
↕	↕	↕	↕
Lennon	McCartney	Harrison	Starr

Example 1 – *Solution*

cont'd

b. Because $n(C) = 4$ and $n(D) = 3$, C and D are not equivalent.

c. A consists of all natural numbers from 1 to 50, inclusive.

Hence, $n(A) = 50$. B consists of all odd natural numbers from 1 to 99, inclusive. Since half of the natural numbers

from 1 to 100 are odd (and half are even), there are fifty ($100 \div 2 = 50$) odd natural numbers less than 100; that is, $n(B) = 50$. Because A and B have the same cardinal number, $A \sim B$.

Example 1 – *Solution*

cont'd

Many different one-to-one correspondences may be established between the elements of *A* and *B*. One such correspondence follows:

A = {1, 2, 3, . . . ,	n,	. . . , 48, 49, 50}
↕ ↕ ↕	↕	↕ ↕ ↕
B = {1, 3, 5, . . . ,	$(2n – 1)$,	. . . , 95, 97, 99}

Countable Sets

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Consider the set of all counting numbers:

$$N = \{1, 2, 3, \ldots\},$$

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Let $O = \{1, 3, 5, . . .\}$ and $E = \{2, 4, 6, . . .\}$

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Can you find $O \cap E$ and $O \cup E$?

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Answer: $O \cap E = \emptyset$ and $O \cup E = N$

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Can you find $O \cap E$ and $O \cup E$?

Answer: $O \cap E = \emptyset$ and $O \cup E = N$

So, the sets O and E are mutually exclusive, and their union forms the entire set of all counting numbers.

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In addition, $O \subset N$, i.e. O is a *proper* subset of N, and $E \subset N$, i.e. E is a *proper* subset of N

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$O \cap E = \emptyset$ hence O and E are mutually exclusive sets, and $O \cup E = N$ so their union forms the entire set of all counting numbers.

In addition, $O \subset N$, i.e. O is a *proper* subset of N, and $E \subset N$, i.e. E is a *proper* subset of N

This might lead people to think that N is "bigger" than E.

In fact, N and E are the "same size": N and E each contain the same number of elements.

Countable Sets

Recall: two sets are equivalent and have the same cardinal number if the elements of the sets can be matched up via a one-to-one correspondence.

So let's find an explicit correspondence between the elements of the two sets: E and N.

Example 2 – *Finding a One-to-One Correspondence Between Two Infinite Sets*

1. Show that *E* = {2, 4, 6, 8, . . .} and *N* = {1, 2, 3, 4, . . .} are equivalent sets.

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1. Show that $E = \{2, 4, 6, 8, . . .\}$ and $N = \{1, 2, 3, 4, . . .\}$ are equivalent sets.

$$N = \{1, 2, 3, 4, . . . , n, . . .\}$$

$$E = \{2, 4, 6, 8, . . . , 2n, . . .\}$$

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 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \ldots & \updownarrow & \\
E = \{ & 2, & 4, & 6, & 8, & \ldots, & 2n, & \ldots\}
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Any natural number $n \in N$ corresponds with the even natural number $2n \in E$.

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↕ ↕ ↕ ↕ . . . ↕

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Hence, $E \sim N$.

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$$E = \{2, 4, 6, 8, . . . , 2n, . . .\}$$

Any natural number $n \in N$ corresponds with the even natural number $2n \in E$.

Hence, $E \sim N$.

This equivalence implies that the two sets have the same number of elements! Although E is a proper subset of N, both sets have the same cardinal number; that is,
n(E) = n(N).

Example 2 – *Finding a One-to-One Correspondence Between Two Infinite Sets*

1. Show that $E = \{2, 4, 6, 8, . . .\}$ and $N = \{1, 2, 3, 4, . . .\}$ are equivalent sets.

2. Can you find the element of N that corresponds to $1430 \in E$?

3. Can you find the element of N that corresponds to $x \in E$?

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2. Can you find the element of N that corresponds to $1430 \in E$?

$1430 = 2n \in E$, so $n = \frac{1430}{2} = 715 \in N$.

Therefore, $715 \in N$ corresponds to $1430 \in E$.

3. Can you find the element of N that corresponds to $x \in E$?

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Therefore, $715 \in N$ corresponds to $1430 \in E$.

3. Can you find the element of N that corresponds to $x \in E$?

$x = 2n \in E$, so $n = \frac{x}{2} \in N$.
Therefore, $n = \frac{x}{2} \in N$ corresponds to $x = 2n \in E$.

Countable Sets

A set is said to be an ***infinite set*** if it can be placed in a one-to-one correspondence with a proper subset of itself.

The *cardinal number* of the set of counting numbers N is defined as $\aleph_0$ (read "**aleph-null").**

Any set that is *equivalent* to the set of counting numbers has cardinal number $\aleph_0$.

A set is ***countable*** if it is finite or if it has cardinality $\aleph_0$.

Example 3 – *Showing that the Set of Integers is Countable*

How can we show that the set *I* of all integers is countable?

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	↕	↕	↕	↕	↕	
I =	{0,	1,	–1,	2,	–2,	. . .}

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- an odd natural number *n* corresponds to a non positive integer $\frac{1-n}{2}$

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$$

What element of *I* corresponds to $613 \in N$?

- an odd natural number *n* corresponds to a non positive integer $\frac{1-n}{2}$
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$$\updownarrow \; \updownarrow \; \updownarrow \; \updownarrow \; \updownarrow$$

$$I = \{0, 1, -1, 2, -2, \ldots\}$$

What element of *I* corresponds to $613 \in N$?

$$\frac{1-613}{2} = -306$$

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What element of *N* corresponds to $853 \in I$?

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What element of *N* corresponds to $853 \in I$?

$$\frac{n}{2} = 853$$

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What element of *N* corresponds to $853 \in I$?

$$\frac{n}{2} = 853 \qquad n = 2 \times 853 = 1706$$

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What element of *N* corresponds to $-397 \in I$?

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What element of *N* corresponds to –397 ∈ *I* ?

$$\frac{1-n}{2} = -397 \qquad 1-n = 2\times(-397)$$

- an odd natural number *n* corresponds to a non positive integer $\frac{1-n}{2}$
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\end{array}
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What element of N corresponds to $-397 \in I$?

$$\frac{1-n}{2} = -397 \qquad 1-n = 2\times(-397)$$

$$n = 795$$

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What is *n(I), the cardinal number* of the set *I ?*

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Some sets, like set $A = \{x |\ 0 \leq x < 1\}$ of real numbers greater than or equal to 0 and less than 1 are *uncountable*; that is, the elements of *A* and *N* cannot be put into a one-to-one correspondence;

We will show it by assuming that they can be put into a one-to-one correspondence and then finding/building a "new" number from A that is not included in this correspondence!

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Let's *assume* that $A = \{x \mid 0 \leq x < 1\}$ is countable!

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That implies that the elements of A and N can be put into a one-to-one correspondence; each $a \in A$ can be listed and counted.

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The elements of *A* are nonnegative real numbers less than 1, each a_n = 0. □ □ □ □ □....

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The elements of *A* are nonnegative real numbers less than 1, each $a_n = 0.\square\square\square\square\square....$

Say, for instance, the numbers in our list are

$a_1 = 0.3750000\ldots$ the first element of *A*

$a_2 = 0.7071067\ldots$ the second element of *A*

$a_3 = 0.5000000\ldots$ the third element of *A*

$a_4 = 0.6666666\ldots$ and so on.

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$a_3 = 0.5000000 \ldots$ the third element of *A*

$a_4 = 0.6666666 \ldots$ and so on.

building *b*:

- if digit is 0, put 1
- otherwise, put 0

Uncountable Sets

Let's *assume* that $A = \{x \mid 0 \le x < 1\}$ is countable!

That implies that the elements of A and N can be put into a one-to-one correspondence; each a ∊ A can be listed and counted.

The elements of *A* are nonnegative real numbers less than 1, each $a_n = 0.\square\square\square\square\square....$

Say, for instance, the numbers in our list are

$a_1 = 0.\underline{3}750000\ldots$ the first element of *A*

$a_2 = 0.7071067\ldots$ the second element of *A*

$a_3 = 0.5000000\ldots$ the third element of *A*

$a_4 = 0.6666666\ldots$ and so on.

building *b*:

- if digit is 0, put 1
- otherwise, put 0

$b = 0.\underline{0}$

Uncountable Sets

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The elements of *A* are nonnegative real numbers less than 1, each a_n = 0.□□□□□....

Say, for instance, the numbers in our list are

a_1 = 0.<u>3</u>750000 . . . the first element of *A*

a_2 = 0.7<u>0</u>71067 . . . the second element of *A*

a_3 = 0.5000000 . . . the third element of *A*

a_4 = 0.6666666 . . . and so on.

building *b*:

- if digit is 0, put 1
- otherwise, put 0

b = 0.<u>01</u>

Uncountable Sets

Let's *assume* that $A = \{x \mid 0 \leq x < 1\}$ is countable!

That implies that the elements of A and N can be put into a one-to-one correspondence; each $a \in A$ can be listed and counted.

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Say, for instance, the numbers in our list are

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$a_3 = 0.50\underline{0}0000 \ldots$ the third element of *A*

$a_4 = 0.6666666 \ldots$ and so on.

building *b*:

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- otherwise, put 0

$b = 0.\underline{011}$

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$a_3 = 0.50\underline{0}0000\ldots$ the third element of *A*

$a_4 = 0.666\underline{6}666\ldots$ and so on.

building *b*:

- if digit is 0, put 1
- otherwise, put 0

$b = 0.\underline{0110}$

Uncountable Sets

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$a_4 = 0.666\underline{6}666 \ldots$

$0 \le b < 1$, hence $b \in A$

building *b*:

- if digit is 0, put 1
- otherwise, put 0

$b = 0.\underline{0110}\ldots$

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That implies that the elements of A and N can be put into a one-to-one correspondence; each a ∊ A can be listed and counted.

The elements of *A* are nonnegative real numbers less than 1, each $a_n = 0.$□□□□□....

Say, for instance, the numbers in our list are

$a_1 = 0.\underline{3}750000 \ldots$

$a_2 = 0.7\underline{0}71067 \ldots$

$a_3 = 0.50\underline{0}0000 \ldots$

$a_4 = 0.666\underline{6}666 \ldots$

$0 \leq b < 1$, hence $b \in A$

$b \neq a_1$, $b \neq a_2$,

$b \neq a_3$, …

building *b*:

- if digit is 0, put 1
- otherwise, put 0

$b = 0.\underline{0110}\ldots$

Uncountable Sets

Let's *assume* that $A = \{x \mid 0 \leq x < 1\}$ is countable!

That implies that the elements of A and N can be put into a one-to-one correspondence; each a ∊ A can be listed and counted.

The elements of *A* are nonnegative real numbers less than 1, each a_n = 0.□□□□□....

Say, for instance, the numbers in our list are

$a_1 = 0.\underline{3}750000 \ldots$

$a_2 = 0.7\underline{0}71067 \ldots$

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$a_4 = 0.666\underline{6}666 \ldots$

$0 \leq b < 1$, hence $b \in A$

$b \neq a_1$, $b \neq a_2$,

$b \neq a_3$, …

building *b*:

- if digit is 0, put 1
- otherwise, put 0

$b = 0.\underline{0110}\ldots$

Hence b is not on our list of all elements of A

Uncountable Sets

This contradicts the assumption that the elements of *A* and *N* can be put into a one-to-one correspondence.

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Since the assumption leads to a contradiction, the assumption must be false; $A = \{x | 0 \leq x < 1\}$ is not countable.

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An infinite set that cannot be put into a one-to-one correspondence with N is said to be ***uncountable*****.**

Uncountable Sets

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Since the assumption leads to a contradiction, the assumption must be false; $A = \{x | 0 \leq x < 1\}$ is not countable.

An infinite set that cannot be put into a one-to-one correspondence with *N* is said to be ***uncountable*****.**

Consequently, an uncountable set has *more* elements than the set of all counting numbers.

Uncountable Sets

This implies that there are different magnitudes of infinity!

To distinguish the magnitude of A from that of N, we can denote cardinality of $A = \{x \mid 0 \le x < 1\}$ as

$$n(A) = c \text{ (} c \text{ for } \textbf{continuum}\text{)}.$$

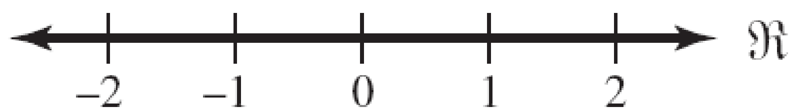
Thus, $\aleph_0 < c$.

It was shown that A is equivalent to the entire set of all real numbers, that is, $A \sim \Re$. Therefore, $n(\Re) = c$.

Points on a Line

Points on a Line

The real number system, denoted by $\Re$, can be put into a one-to-one correspondence with all points on a line, such that every real number corresponds to exactly one point on a line and every point on a line corresponds to exactly one real number. Consequently, any (infinite) line contains *c* points.



The real number line.

Figure 2.45

Example 5 – *Showing that Line Segments of Different Lengths are Equivalent Sets of Points*

Show that the line segments [0, 1] and [0, 2] are equivalent sets of points.

Solution:

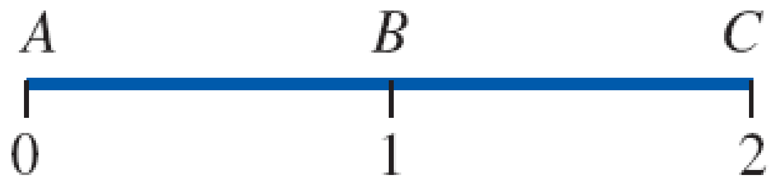
Because the segment [0, 2] is twice as long as the segment [0, 1], intuition might tell us that it contains twice as many points. Not so!

Two sets are equivalent (and have the same cardinal number) if their elements can be put into a one-to-one correspondence.

Example 5 – *Solution*

cont'd

On a number line, let *A* represent the point 0, let *B* represent 1, and let *C* represent 2, as shown in Figure 2.46.



The interval [0, 2].

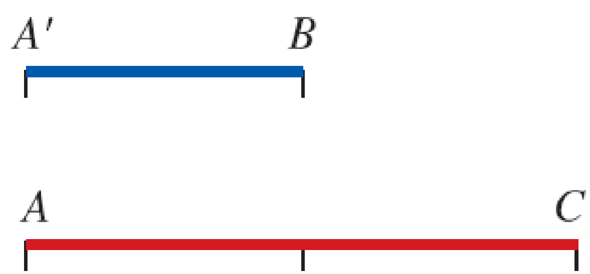
Figure 2.46

Example 5 – *Solution*

cont'd

Our goal is to develop a one-to-one correspondence between the elements of the segments *AB* and *AC.*

Now draw the segments separately, with *AB* above *AC,* as shown in Figure 2.47. (To distinguish the segments from each other, point *A* of segment *AB* has been relabeled as point *A′*.)



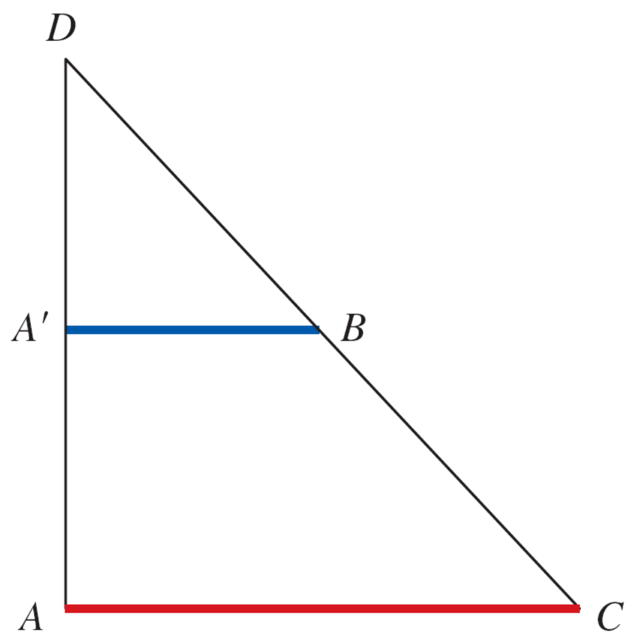
The intervals [*A*, *B*] and [*A*, *C*].

Figure 2.47

Example 5 – *Solution*

cont'd

Extend segments AA' and CB so that they meet at point D, as shown in Figure 2.48.



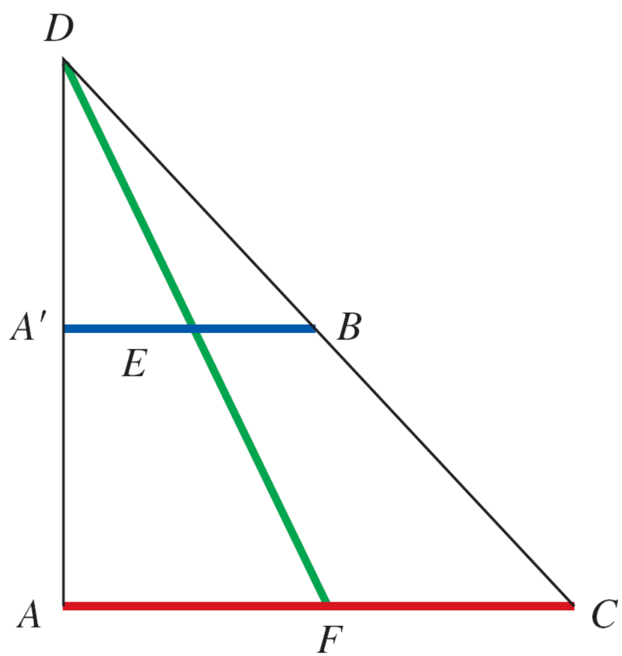
Extending AA' and CB to form point D.

Figure 2.48

Example 5 – *Solution*

cont'd

Any point E on $A'B$ can be paired up with the unique point F on AC formed by the intersection of lines DE and AC, as shown in Figure 2.49.



A one-to-one correspondence between line segments.

Figure 2.49

Example 5 – *Solution*

cont'd

Conversely, any point *F* on segment *AC* can be paired up with the unique point *E* on *A′B* formed by the intersection of lines *DF* and *A′B.*

Therefore, a one-to-one correspondence exists between the two segments, so [0, 1] ~ [0, 2].

Consequently, the interval [0, 1] contains exactly the same number of points as the interval [0, 2]!