2

Sets and Counting

2.3 Introduction to Combinatorics

Objectives

- Develop and apply the Fundamental Principle of Counting

- Develop and evaluate factorials

Introduction to Combinatorics

If you went on a shopping spree and bought two pairs of jeans, three shirts, and two pairs of shoes, how many new outfits (consisting of a new pair of jeans, a new shirt, and a new pair of shoes) would you have?

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A compact disc buyers’ club sends you a brochure saying that you can pick any five CDs from a group of 50 of today’s hottest sounds for only $19.95.How many different combinations can you choose?

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Six local bands have volunteered to perform at a benefit concert, and there is some concern over the order in which the bands will perform. How many different lineups are possible?

Introduction to Combinatorics

The answers to questions like these can be obtained by listing all the possibilities or by using three shortcut counting methods: the **Fundamental Principle of Counting**, **combinations**, and **permutations**.

Collectively, these methods are known as ***combinatorics***. (Incidentally, the answers to the questions above are 12 outfits, 2,118,760 CD combinations, and 720 lineups.) Let us consider the first shortcut method.

The Fundamental Principle of Counting

The Fundamental Principle of Counting

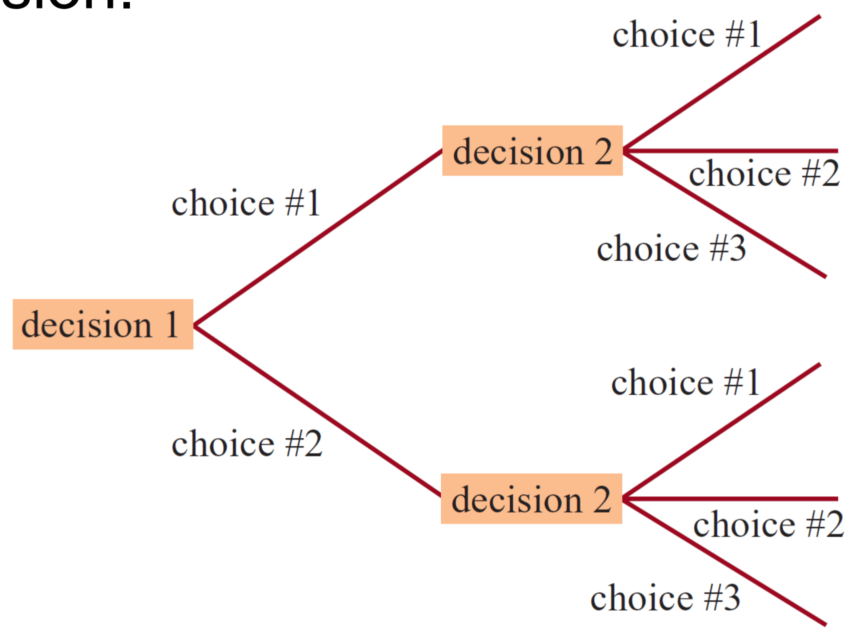
Daily life requires that we make many decisions.

When making a series of decisions, how can you determine the total number of possible selections? One way is to list all the choices for each category and then match them up in all possible ways.

To ensure that the choices are matched up in all possible ways, you can construct a ***tree diagram***.

The Fundamental Principle of Counting

A tree diagram consists of clusters of line segments, or *branches*, constructed as follows: a cluster of branches is drawn for each decision to be made such that the number of branches in each cluster equals the number of choices for the decision.



A tree diagram.

Figure 2.37

The Fundamental Principle of Counting

Although this method can be applied to all problems, it is very time consuming and impractical when you are dealing with a series of many decisions, each of which contains numerous choices.

Instead of actually listing all possibilities via a tree diagram, using a shortcut method might be desirable. The following example gives a clue to finding such a shortcut.

Example 1 – *Determining the Total Number of Possible Choices in a Series of Decisions*

If you buy two pairs of jeans, three shirts, and two pairs of shoes, how many new outfits (consisting of a new pair of jeans, a new shirt, and a new pair of shoes) would you have?

Example 1 – *Determining the Total Number of Possible Choices in a Series of Decisions*

If you buy two pairs of jeans, three shirts, and two pairs of shoes, how many new outfits (consisting of a new pair of jeans, a new shirt, and a new pair of shoes) would you have?

Solution:

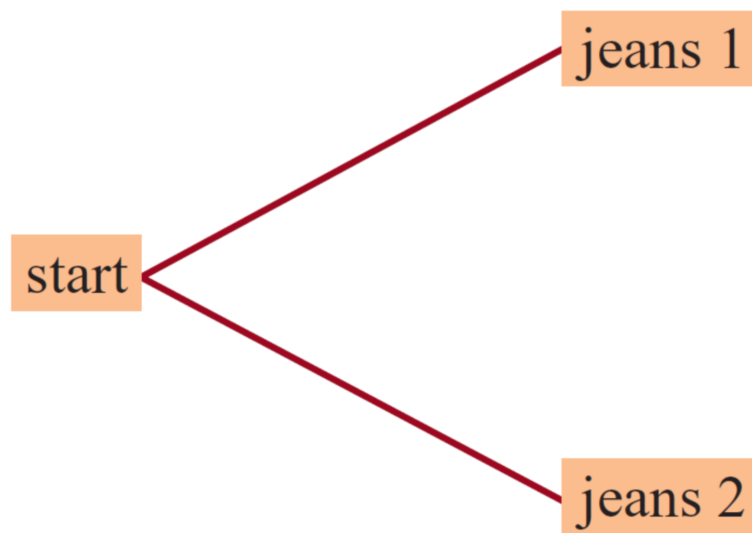
Because there are three categories, selecting an outfit requires a series of three decisions: You must select one pair of jeans, one shirt, and one pair of shoes.

We will make our three decisions in the following order: jeans, shirt, and shoes. (The order in which the decisions are made does not affect the overall outfit.)

Example 1 – *Solution*

cont'd

Our first decision (jeans) has two choices (jeans 1 or jeans 2); our tree starts with two branches, as in Figure 2.38.



The first decision.

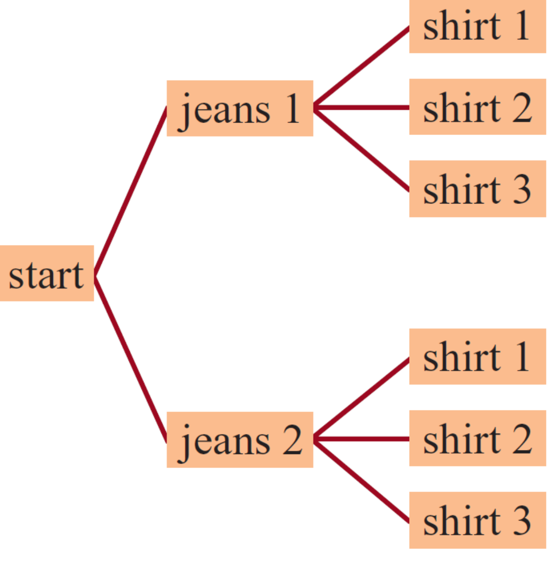
Figure 2.38

Example 1 – *Solution*

cont'd

Our second decision is to select a shirt, for which there are three choices.

At each pair of jeans on the tree, we draw a cluster of three branches, one for each shirt, as in Figure 2.39.



The Second decision.

Figure 2.39

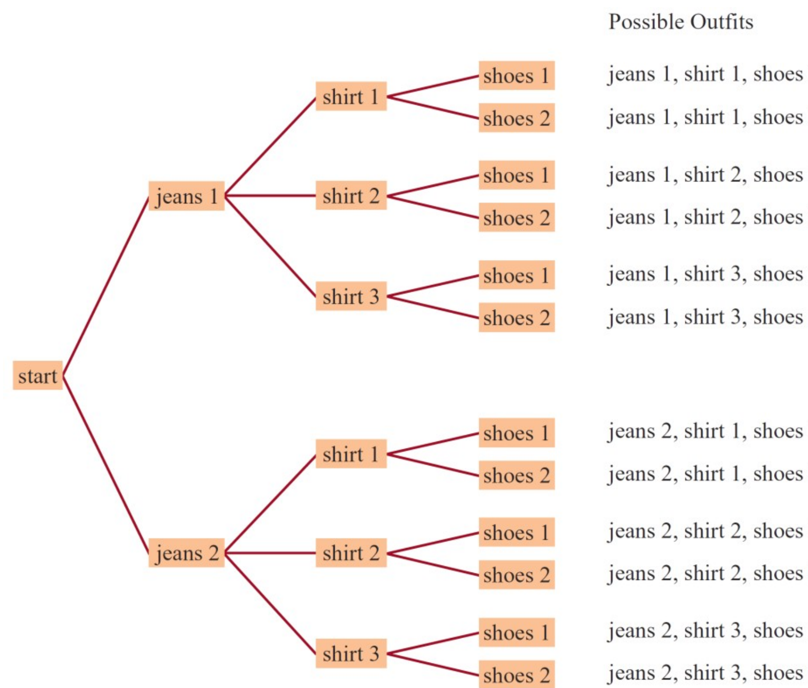
Example 1 – *Solution*

cont'd

Our third decision is to select a pair of shoes, for which there are two choices.

At each shirt on the tree, we draw a cluster of two branches, one for each pair of shoes, as in Figure 2.40.

We have now listed all possible ways of putting together a new outfit; twelve outfits can be formed from two pairs of jeans, three shirts, and two pairs of shoes.



The Third decision.

Figure 2.40

The Fundamental Principle of Counting

Going back to question: *If you buy two pairs of jeans, three shirts, and two pairs of shoes, how many new outfits would you have?*

Did you notice that for when making a decision, <u>the number of branches on the tree diagram was *multiplied* by a factor equal to the number of choices for the decision</u> ?

The Fundamental Principle of Counting

Going back to question: *If you buy two pairs of jeans, three shirts, and two pairs of shoes, how many new outfits would you have?*

Did you notice that for when making a decision, the number of branches on the tree diagram was *multiplied* by a factor equal to the number of choices for the decision ?

Therefore, the total number of outfits could have been obtained by *multiplying* the number of choices for each decision:

$$\underset{\text{jeans}}{2}\cdot\underset{\text{shirts}}{3}\cdot\underset{\text{shoes}}{2}=\underset{\text{outfits}}{12}$$

The Fundamental Principle of Counting

The generalization of this process of multiplication is called the *Fundamental Principle of Counting*.

The Fundamental Principle of Counting

The total number of possible outcomes of a series of decisions (making selections from various categories) is found by multiplying the number of choices for each decision (or category) as follows:

1. Draw a box for each decision.
2. Enter the number of choices for each decision in the appropriate box and multiply.

Example 2 – *Applying the Fundamental Principle of Counting*

A serial number consists of two consonants followed by three nonzero digits followed by a vowel (A, E, I, O, U): for example, “ST423E” and “DD666E.” Determine how many serial numbers are possible given the following conditions.

a. Letters and digits cannot be repeated in the same serial number.

b. Letters and digits can be repeated in the same serial number.

Example 2 – *Solution*

a. Letters and digits cannot be repeated in the same serial number. Example: “ST423E”

Let’s draw six boxes for the serial number :

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Let’s draw six boxes for the serial number :

$$\boxed{21}\quad\boxed{}\quad\boxed{}\quad\boxed{}\quad\boxed{}\quad\boxed{}$$

There are 21 different choices for the first consonant.

Example 2 – *Solution*

a. Letters and digits cannot be repeated in the same serial number. Example: “ST423E”

Let’s draw six boxes for the serial number :

$\boxed{21}\ \boxed{20}\ \boxed{}\ \boxed{}\ \boxed{}\ \boxed{}$

There are 21 different choices for the first consonant.

Because the letters cannot be repeated, there are only 20 choices for the second consonant.

Example 2 – *Solution*

a. Letters and digits cannot be repeated in the same serial number. Example: “ST423E”

Let’s draw six boxes for the serial number :

21	20	9	8	7	

There are 21 different choices for the first consonant.

Because the letters cannot be repeated, there are only 20 choices for the second consonant.

There are 9 different choices for the first nonzero digit, 8 choices for the second, and 7 choices for the third.

Example 2 – *Solution*

a. Letters and digits cannot be repeated in the same serial number. Example: “ST423E”

Let’s draw six boxes for the serial number :

21	20	9	8	7	5

There are 5 different vowels: A, E, I, O, U

There are 21 different choices for the first consonant.

Because the letters cannot be repeated, there are only 20 choices for the second consonant.

There are 9 different choices for the first nonzero digit, 8 choices for the second, and 7 choices for the third.

Example 2 – *Solution*

a. Letters and digits cannot be repeated in the same serial number. Example: “ST423E”

Let’s draw six boxes for the serial number :

$$\boxed{21} \quad \boxed{20} \quad \boxed{9} \quad \boxed{8} \quad \boxed{7} \quad \boxed{5}$$

$$\underbrace{\boxed{21} \times \boxed{20}}_{\text{consonants}} \times \underbrace{\boxed{9} \times \boxed{8} \times \boxed{7}}_{\text{nonzero digits}} \times \underbrace{\boxed{5}}_{\text{vowel}} = 1{,}058{,}400$$

There are 1,058,400 possible serial numbers when the letters and digits cannot be repeated within a serial number.

Example 2 – *Solution*

cont'd

b. Letters and digits can be repeated in the same serial number.

Example 2 – *Solution*

b. Letters and digits can be repeated in the same serial number.

Hence, the number of choices does not decrease by one each time as in part (a).

Example 2 – *Solution*

cont'd

b. Letters and digits can be repeated in the same serial number.

Hence, the number of choices does not decrease by one each time as in part (a).

The total number of possibilities is

$$\underbrace{\boxed{21} \times \boxed{21}}_{\text{consonants}} \times \underbrace{\boxed{9} \times \boxed{9} \times \boxed{9}}_{\text{nonzero digits}} \times \underbrace{\boxed{5}}_{\text{vowel}} = 1{,}607{,}445$$

There are 1,607,445 possible serial numbers when the letters and digits can be repeated within a serial number.

Practice

Exercise 1: A die is rolled, and a coin is tossed. How many different outcomes are possible?

Practice

Exercise 1: A die is rolled, and a coin is tossed. How many different outcomes are possible?

Solution: a die has 6 sides: 1, 2, 3, 4, 5, and 6 dots

There are two sides of coin: heads and tails

Therefore,

6	2

$6 * 2 = 12$

Answer: 12 different outcomes are possible

Practice

Exercise 2: Consider the following requirements for a password to an email account:

- it should consist of 7 characters.
- each of these characters must be either a digit or a letter of the English alphabet (lower case, upper case)
- each password must start with a letter of the alphabet and must end with a digit
- Neither digits nor letters of alphabet (with their case) repeat
- How many different passwords are there?

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- How many different passwords are there?

Hint:

52						10

There are 10 digits, 26(lower case) + 26 (upper case) letters of the alphabet – fix the first and the last characters

Practice

Exercise 2: Consider the following requirements for a password to an email account:

- it should consist of 7 characters.
- each of these characters must be either a digit or a letter of the English alphabet (lower case, upper case)
- each password must start with a letter of the alphabet and must end with a digit
- Neither digits nor letters of alphabet (with their case) repeat
- How many different passwords are there?

Solution:

52	60	59	58	57	56	10

There are 9 digits, 26(lower case) + 26 (upper case) letters of the alphabet; second slot: 52-1 + 10-1 = 60

52 * 60 * 59 * 58 * 57 * 56 * 10 = 340,798,348,800 possible passwords

Factorials

Example 3 – *Applying the Fundamental Principle of Counting*

Three students rent a three-bedroom house near campus. One of the bedrooms is very desirable (it has its own bath), one has a balcony, and one is undesirable (it is very small). In how many ways can the housemates choose the bedrooms?

Example 3 – *Applying the Fundamental Principle of Counting*

Three students rent a three-bedroom house near campus. One of the bedrooms is very desirable (it has its own bath), one has a balcony, and one is undesirable (it is very small). In how many ways can the housemates choose the bedrooms?

Solution:

Three decisions must be made: who gets the room with the bath, who gets the room with the balcony, and who gets the small room.

Hence, we should have three boxes.

Example 3 – *Solution*

cont'd

Using the Fundamental Principle of Counting, we draw three boxes and enter the number of choices for each decision.

There are three choices for who gets the room with the bath. Once that decision has been made, there are two choices for who gets the room with the balcony, and finally, there is only one choice for the small room.

$$\boxed{3} \times \boxed{2} \times \boxed{1} = 6$$

There are six different ways in which the three housemates can choose the three bedrooms.

Factorials

Combinatorics often involve products of the type $3 \cdot 2 \cdot 1 = 6$, as seen in Example 3. This type of product is called a **factorial**, and the product $3 \cdot 2 \cdot 1$ is written as $3!$. In this manner,

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 1 \cdot 2 \cdot 3 \cdot 4 = (= 24)$,

and $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \ (= 120)$.

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and $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ $(= 120)$.

Factorials

If n is a positive integer, then n *factorial,* denoted by $n!$, is the product of all positive integers less than or equal to n.

$$n! = n \cdot (n - 1) \cdot (n - 2) \cdot \cdots \cdot 2 \cdot 1$$

As a special case, we define $0! = 1$.

Example 4 – *Evaluating Factorials*

Find the following values.

a. $6!$ **b.** $\dfrac{8!}{5!}$ **c.** $\dfrac{8!}{3! \cdot 5!}$

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Solution:

a. $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$= 720$

Therefore, $6! = 720$.

Example 4 – *Solution*

cont'd

b. $$\frac{8!}{5!}=\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1}=8\cdot7\cdot6=336$$

Therefore, $\frac{8!}{5!} = 336$.

c. $$\frac{8!}{3!5!}=\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{(3\cdot2\cdot1)(5\cdot4\cdot3\cdot2\cdot1)}=\frac{8\cdot7\cdot6}{3\cdot2\cdot1}=56$$

Therefore, $\frac{8!}{3!\cdot5!} = 56$

Practice

Exercise 1: Evaluate the factorial expressions:

(a) 0!

(b) 2!

(c) $\dfrac{6!\,30!}{35!}$

Practice

Exercise 1: Evaluate the factorial expressions:

(a) 0! **(b)** 2! **(c)** $\frac{6!\,30!}{35!}$

Solution:

(a) 0! = 1 by definition

(b) 2! = 2 * 1 = 2

(c) $$\frac{6!\,30!}{35!} = \frac{6!}{35\cdot34\cdot33\cdot32\cdot31} = \frac{\overset{3}{\cancel{6}}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot1}{\underset{7}{\cancel{35}}\cdot\underset{17}{\cancel{34}}\cdot\underset{11}{\cancel{33}}\cdot\underset{\underset{4}{\cancel{8}}}{\cancel{32}}\cdot31} = \frac{3}{7\cdot17\cdot11\cdot4\cdot31} =$$

$$= \frac{3}{162316} \approx 0.0000062$$