CS135 Midterm Exam Sample

N1

$$\sum_{i=12}^{i=128} 17 = (128-12+1)\cdot 17 = 117 \cdot 17 = \boxed{1{,}989}$$

use formula $\sum_{i=m}^{n} c = (n-m+1)c$

N2

$$\sum_{i=5}^{9} (-1)^i \frac{(i-7)^2}{10} = \left(-\frac{4}{10}\right) + \left(\frac{1}{10}\right) + 0 + \left(\frac{1}{10}\right) + \left(-\frac{4}{10}\right) =$$

$i=5: \quad (-1)^5 \frac{(5-7)^2}{10} = -\frac{4}{10}$

$i=6: \quad (-1)^6 \frac{(6-7)^2}{10} = \frac{1}{10}$

$i=7: \quad (-1)^7 \frac{(7-7)^2}{10} = 0$

$i=8: \quad (-1)^8 \frac{(8-7)^2}{10} = \frac{1}{10}$

$i=9: \quad (-1)^5 \frac{(9-7)^2}{10} = -\frac{4}{10}$

$$= -\frac{6}{10} = \boxed{-\frac{3}{5}}$$

N3 $\quad 3 + (-12) + 48 + \dots + 12{,}288$

1) $3 \cdot (-4) = -12$
$-12 \cdot (-4) = 48$
therefore $3, -12, 48, \dots,$ is a geometric progressien/sequence with $r = (-4)$ and $a_0 = 3$

2) we know that $\sum_{i=0}^{n-1} a_0 r^i = \frac{a_0(r^n - 1)}{r-1}$

3) let's find $n-1$: $12{,}288 = 3 \cdot (-4)^{?}$ $\quad 4{,}096 = (-4)^x$

$x = 6$

So $a_0 = 3$, $a_1 = -12$, $a_3 = 48$, $\dots$, $a_6 = 12{,}288$, hence $n-1 = 6$

$$\sum_{i=0}^{6} \underset{3}{a_0}\, \underset{(-4)}{r}^i = \frac{\overset{3}{a_0}(r^7-1)}{\underset{(-4)}{r}-1} = \frac{3((-4)^7-1)}{(-4)-1} = \frac{3(-16385)}{-5} = \boxed{9831}$$

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N4. $9 \mid n^3 + (n+1)^3 + (n+2)^3 \leftarrow P(n)$, for all $n \in \mathbb{Z}$, $n \geq 0$

Proof: Basis step: let's show that $P(0)$ is true:

$0^3 + (0+1)^3 + (0+2)^3 = 1 + 8 = 9 \qquad 9 \mid 9$

Inductive step: assume that for any arbitrary fixed $k \geq 0$, $P(k)$ is true, i.e. $9 \mid k^3 + (k+1)^3 + (k+2)^3$. (IH)

Let's show that in this case $P(k+1)$: $9 \mid (k+1)^3 + (k+2)^3 + (k+3)^3$ is also true:

$$(k+1)^3 + (k+2)^3 + (k+3)^3 = (k+1)^3 + (k+2)^3 + k^3 + 3 \cdot k^2 \cdot 3 + 3 \cdot k \cdot 3^2 + 3^3 =$$

$$= k^3 + (k+1)^3 + (k+2)^3 + 9k^2 + 27k + 27 =$$

$$= \underbrace{\underbrace{k^3 + (k+1)^3 + (k+2)^3}_{\text{divisible by 9 by (IH)}} + \underbrace{9(k^2 + 3k + 3)}_{\text{divisible by 9}}}_{\text{divisible by 9 by theorem}}$$

Therefore, we ~~demonstrated~~ showed that $9 \mid (k+1)^3 + (k+2)^3 + (k+3)^3$, i.e. $P(k+1)$ is true

This completes the inductive step.

By mathematical induction we proved that $9 \mid n^3 + (n+1)^3 + (n+2)^3$ for all non-negative integers n.

q.e.d.

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N5 $\quad f : \mathbb{Z}^+ \to \mathbb{Z}$.

1) the given definition is valid, because $f(n) = 2f(n-3)$ is defined for all $n \geq 4$

2) $f(1) = 1$, $\quad f(2) = 0$, $\quad f(3) = 2$,

$f(4) = 2 \cdot f(1) = 2 \cdot 1 = 2$, $\quad f(5) = 2 \cdot f(2) = 2 \cdot 0 = 0$,

$f(6) = 2 \cdot f(3) = 2 \cdot 2 = 4$, $\quad f(7) = 2 \cdot f(4) = 2 \cdot 2 = 4$

$f(8) = 2 \cdot f(5) = 2 \cdot 0 = 0$, $\quad f(9) = 2 \cdot f(6) = 2 \cdot 4 = 8$

$f(10) = 2f(7) = 2 \cdot 4 = 8$, ...

or 3| (n+1)
(as we discussed in class)

$$\boxed{f(n) = \begin{cases} 0, & \text{if } n \bmod 3 = 2 \\ 2^{n \operatorname{div} 3}, & \text{otherwise} \end{cases}}$$

comment: $n \operatorname{div} 3$ is quotient from division $n \div 3$.

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N6 $P_m(n) = m \cdot n$, $m \in \mathbb{Z}$, $n \in \mathbb{Z}^+ \cup \{0\}$

$$m \cdot n = \underbrace{m + m + m + \dots + m}_{n \ m\text{'s}}$$

$$\boxed{P_m(n) = \begin{cases} 0, & \text{if } m = 0 \\ m + P_m(n-1), & \text{if } n \neq 0 \end{cases}}$$

Let's check: $P_3(5) = 3 + P_3(4) =$

$= 3 + 3 + P_3(3) =$

$= 3 + 3 + 3 + P_3(2) =$

$= 3 + 3 + 3 + 3 + P_3(1) =$

$= 3 + 3 + 3 + 3 + 3 + P_3(0) =$

$= 3 + 3 + 3 + 3 + 3 + 0 = 3 \cdot 5 = 15$

$= 3 \cdot 5 = 15$

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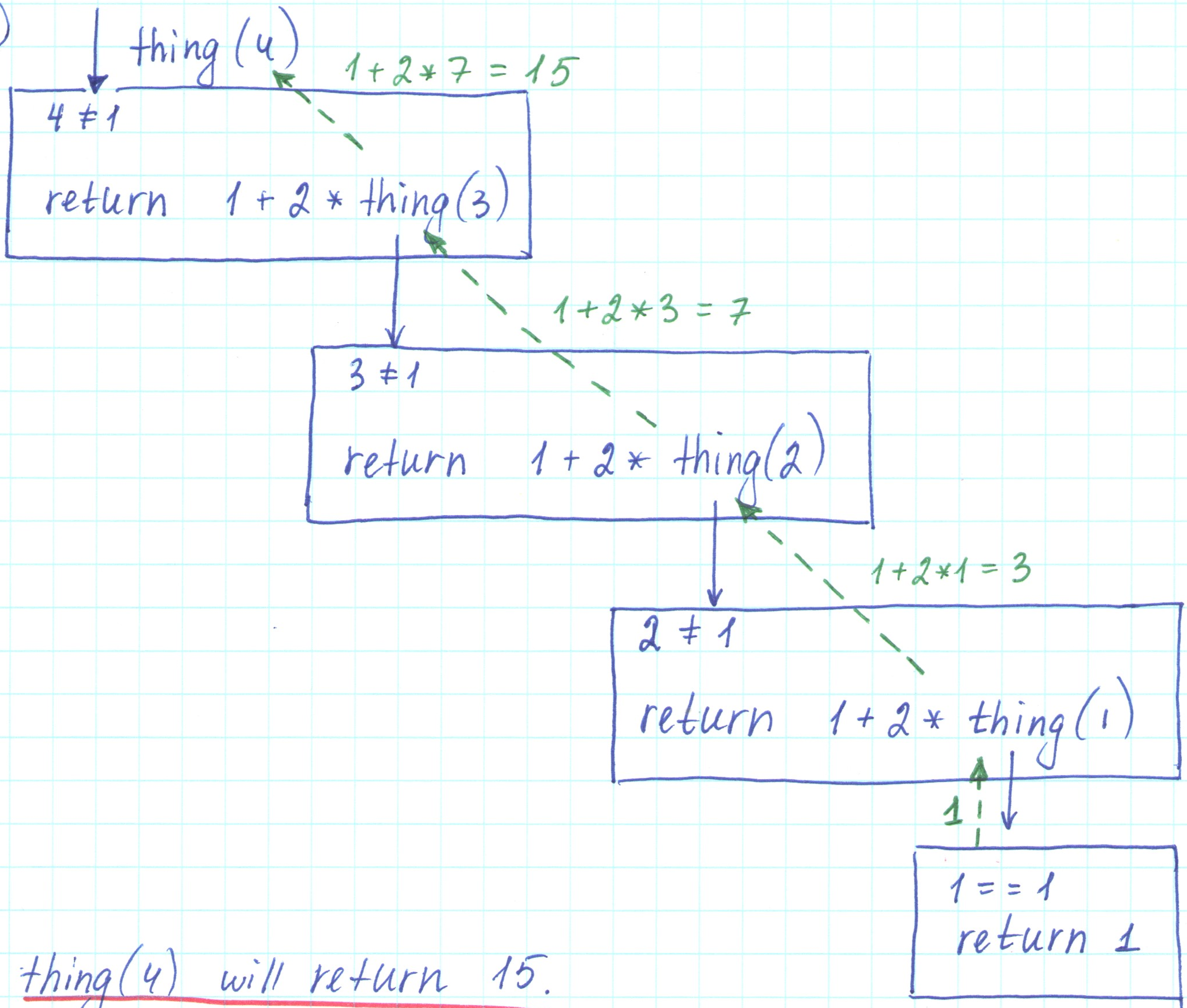
N7



thing(4) will return 15.

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N8 $\quad R = \{ (a,b) \mid a \equiv b \pmod 7 \}$ on $\overset{S}{\{1,2,3,8,10,15\}}$

a) $1 \equiv 8 \pmod 7 \qquad 8 \equiv 15 \pmod 7 \qquad 1 \equiv 15 \pmod 7$

and vice versa

$2 \equiv ?$ nothing $\qquad 2 \equiv 2 \pmod 7$

$3 \equiv 10 \pmod 7$

Therefore, $\boxed{R = \{ (1,8), (8,1), (8,15), (15,8), (1,15), (15,1), (3,10), (10,3), (1,1), (2,2), (3,3), (8,8), (10,10), (15,15) \}}$

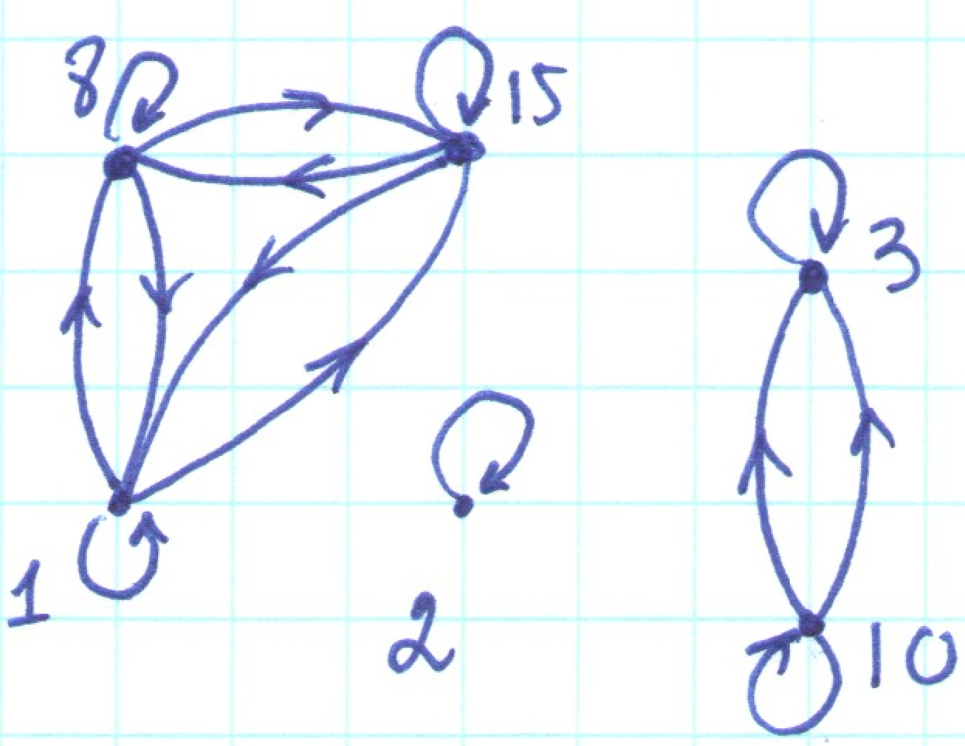
14 pairs

b)

$$\boxed{\begin{array}{c|cccccc} & 1 & 2 & 3 & 8 & 10 & 15 \\ \hline 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 1 & 0 \\ 8 & 1 & 0 & 0 & 1 & 0 & 1 \\ 10 & 0 & 0 & 1 & 0 & 1 & 0 \\ 15 & 1 & 0 & 0 & 1 & 0 & 1 \end{array} = M_R}$$

14 $\quad$ 115

c)



d) • R is reflexive, because $(x,x) \in R$ for all $x \in S$

• R is symmetric, because if $(x,y) \in R$ then $(y,x) \in R$, $x,y \in S$, $x \neq y$

• R is not antisymmetric, because $(1,8), (8,1) \in R$ but $1 \neq 8$

• R is not asymmetric, because both $(1,8)$ and $(8,1) \in R$

• R is transitive, because $\underset{\in R}{(1,8)}$ & $\underset{\in R}{(8,15)} \to \underset{\in R}{(1,15)}$, $\underset{\in R}{(1,15)}$ & $\underset{\in R}{(15,8)} \to \underset{\in R}{(1,8)}$ and so ~~forth~~ forth.

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N9

$$M_S = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

a) S^{-1} is the inverse of S, i.e. for every (x,y) from S, S^{-1} will have (y,x)

$$M_S = \begin{bmatrix} 0 & 1^{0} & 1 \\ {}^{1}0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{S^{-1}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

b) Find $\bar{S}$. $\bar{S}$ is all pairs $(x,y) \notin S$, $x, y \in A$

$$M_S = \begin{bmatrix} \cancel{0}\,1 & \cancel{1}\,0 & \cancel{1}\,0 \\ \cancel{0}\,1 & \cancel{0}\,1 & \cancel{1}\,0 \\ \cancel{1}\,0 & \cancel{1}\,0 & \cancel{0}\,1 \end{bmatrix}$$

$$M_{\bar{S}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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N10

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

a) $R \cup S$ - union of R and S

(put 1 if it is in M_R or in M_S)

$$\boxed{M_{R\cup S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}}$$

b) $R \cap S$ - intersection of R and S

(put 1 only if it is in both, M_R and M_S)

$$\boxed{M_{R\cap S} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$$

c) $S \circ R$ - ~~binary~~ composition of S and R

(binary product of zero-one matrices) order is important!

$$\underset{M_S}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \odot \underset{M_R}{\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}} = \begin{bmatrix} 1\wedge 0 \vee 1\wedge 0 \vee 0\wedge 1 & 1\wedge 1 \vee \ldots & 1\wedge 1 \vee \ldots \\ 1\wedge 0 \vee 1\wedge 0 \vee 1\wedge 1 & 1\wedge 1 \vee \ldots & 1\wedge 1 \vee \ldots \\ 0\wedge 0 \vee 1\wedge 0 \vee 0\wedge 1 & 0\wedge 1 \vee 1\wedge 0 \vee 0\wedge 1 & 0\wedge 1 \vee 1\wedge 1 \ldots \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = M_{S\circ R}$$

d) $S - R$ remove all 1's that are present in both M_R and M_S from M_S, keep all zeros

$$\boxed{M_{S-R} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

order is important!

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N12 a) the given relation is not an equivalence relation because it is not symmetric: (1,4) is present, but (4,1) is not.

b) the given relation is a partial order, because it is

- reflexive $(1,1), (2,2), (3,3), (4,4) \in R$
- antisymmetric no pairs (a,b) and (b,a), where $a \neq b$, are present.
- transitive $(1,4), (4,3) \to (1,3)$ ✓ present
  $(1,3), (3,2) \to (1,2)$ ✓ present
  $(4,3), (3,2) \to (4,3)$ X $(4,3)$ is not present.

Therefore, the given relation is not a partial order.

N11 $S =$ bit strings of length 7

$R = \{ (x,y) \mid x$ and y agree in their last 4 bits $\}$

- reflexive, because $(x,x) \in R$, i.e. same strings do agree on the last 4 bits
- symmetric, because if $(x,y) \in R$ then obviously $(y,x) \in R$
- transitive, because if two bit-strings x and y agree in their last 4 bits, and y and z agree in their last 4 bits, then obviously x and z agree in their last 4 bits.

Therefore, R is an equivalence relation.

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N13 Set $\{a, b, c, d, e, f\}$.

partition $\{a\}$, $\{b, c, d\}$, and $\{e, f\}$ create:

$\{a\}$: (a,a)

$\{b,c,d\}$: $(b,b), (b,c), (b,d), (c,b), (c,c), (c,d), (d,b), (d,c), (d,d)$

$\{e,f\}$: $(e,e), (e,f), (f,e), (f,f)$

Therefore, $\boxed{R = \{(a,a), (b,b), (b,c), (b,d), (c,b), (c,c), (c,d), (d,b), (d,c), (d,d), (e,e), (e,f), (f,e), (f,f)\}}$

N14 is $(\mathbb{Z}, <)$ a poset?

- reflexive? no $(a,a) \in$ "$<$", because $a \not< a$

Therefore <u>it is not a poset.</u>

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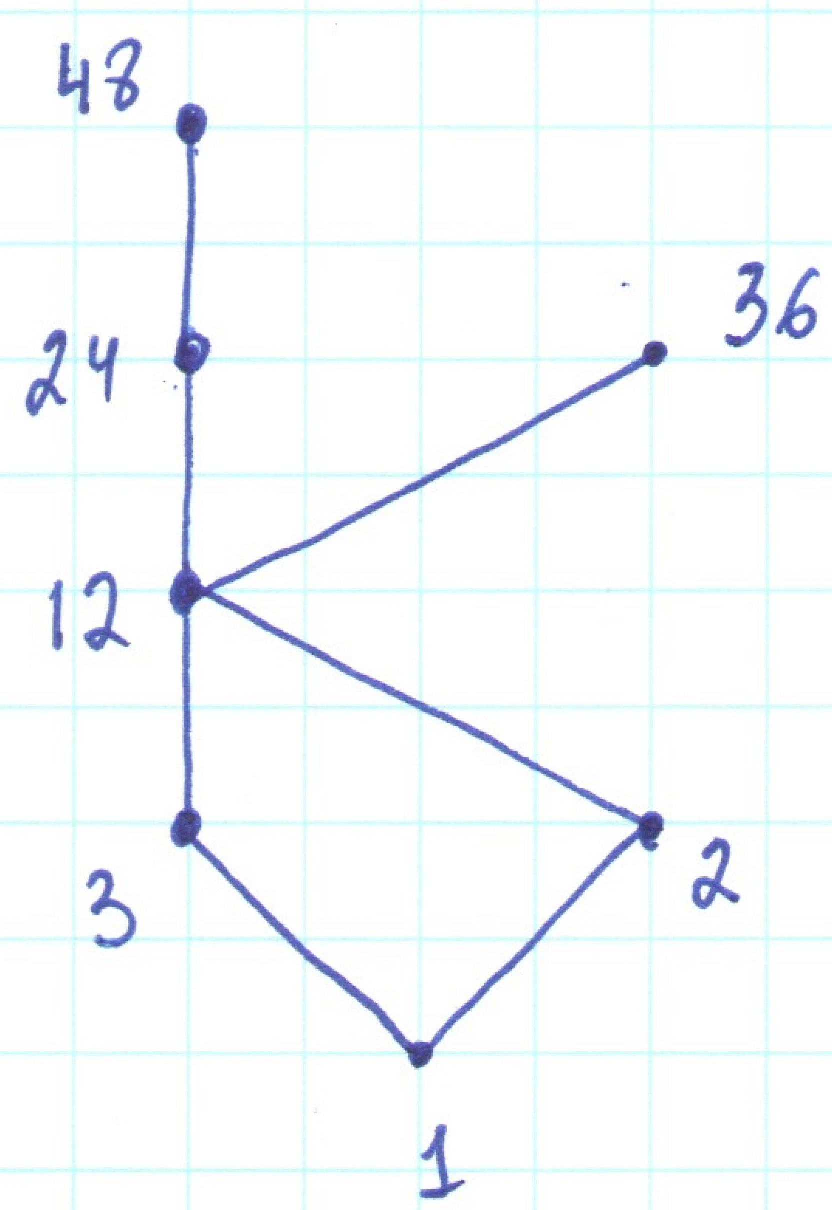
N15 $\{1, 2, 3, 4, 12, 24, 36, 48\}$ divisibility

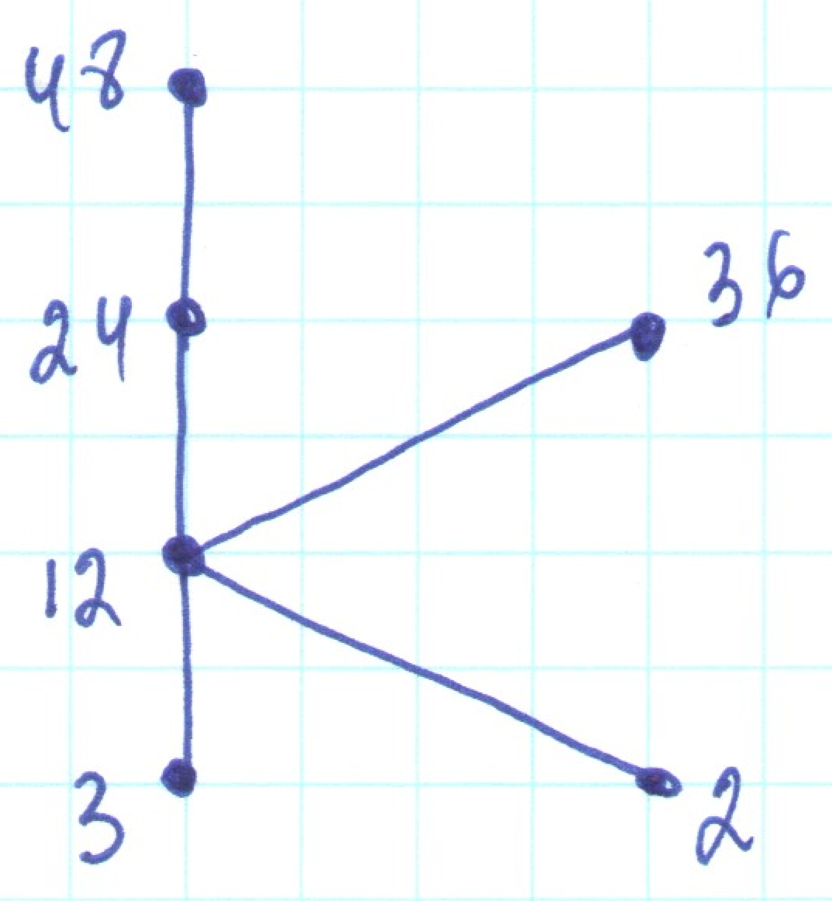
1| everything
2| 4, 12, 24, 36, 48, 2
3| 12, 24, 36, 48, 3
12| 24, 36, 48, 12
24| 48, 24
36| ~~noth~~ 36
48| 48



Hasse diagram

maximal elements: 48, 36
minimal element: 1
greatest element: none
least element: 1
upper bounds of $\{2,3\}$: 12, 24, 36, 48
lower bounds of $\{12, 24\}$ 12, 3, 2, 1

Note: assume that 1 was excluded from the set, then Hasse diagram would be:



then there is no least element
minimal elements: 2, 3