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# Section 9.6 *Partial orderings*

**Convention:**
In different *posets*, different symbols are used for partial ordering ($\leq$, $\geq$, $\supseteq$, $\subseteq$, $|$).
The notation $\preccurlyeq$ is used to denote that $(a,b) \in R$ in an arbitrary *poset* $(S,R)$

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c) *5* and *13*

*5 and 13 are* incomparable because neither *5* | *13* nor *13* | *5*

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**Example 1**: The poset $(S, \geq)$ is totally ordered because $a \geq b$ or $b \geq a$, whenever $a$ and $b$ are integers.

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**Example 2**: The poset $(S, |)$ is not totally ordered. For example $5 \nmid 13$ and $13 \nmid 5$, i.e. 5 and 13 are incomparable.

# Section 9.6 *Partial orderings*

**[Def]** $(S, \preccurlyeq)$ is a *well-ordered set* if it is a *poset* such that $\preccurlyeq$ is a *total ordering* and every non-empty subset of $S$ has at least one element.

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1) the set of of integers **Z**, with the usual $\leq$ ordering, is *not well-ordered*, because the set of negative integers is a subset of **Z**, but doesn't have the smallest element.

2) The set of positive integers $\mathbf{Z}^+$, with the usual $\leq$ ordering, is *well-ordered*.

# Section 9.6 *Partial orderings*

**[Theorem]** The principle of well-ordered induction

Suppose that S is a *well-ordered set*.

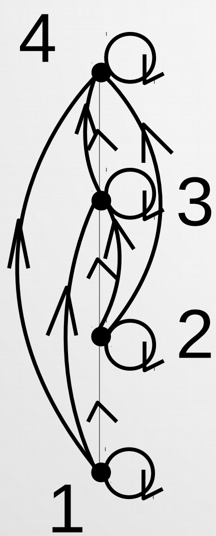
Then P(x) is true for $\forall\ x \in S$, if

Inductive step: For every $y \in S$, if P(x) is true for $\forall\ x \in S$ with $x \prec y$, then P(y) is true.

# Section 9.6 *Partial orderings*

## Hasse Diagrams

Consider the partial ordering $\{(a,b) \mid a \leq b\}$ on the set $S = \{1,2,3,4\}$. Let's present a poset $(S, \preccurlyeq)$ using directed graphs:



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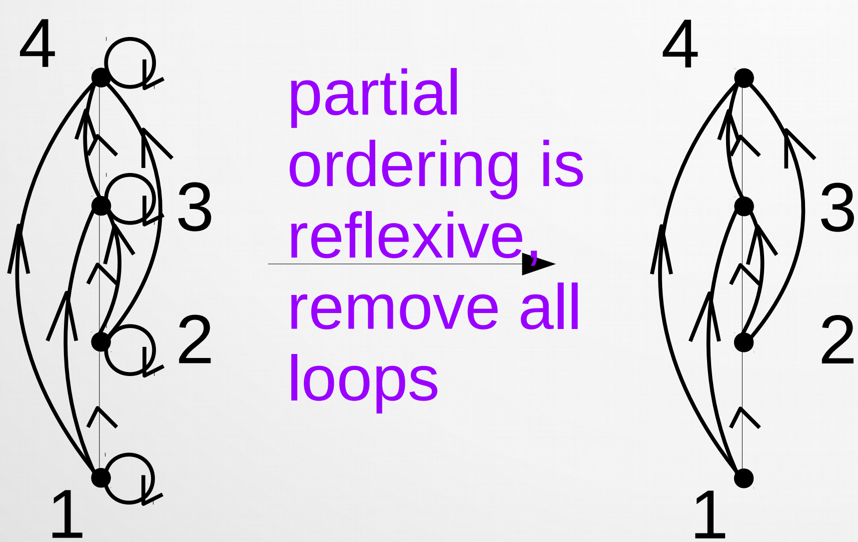

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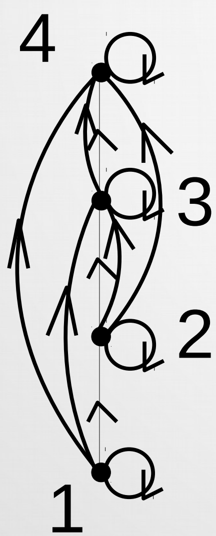
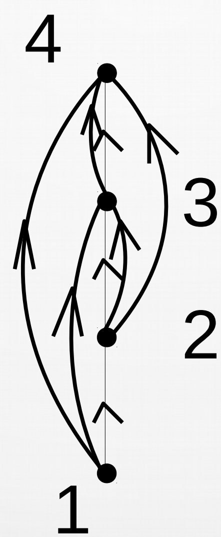
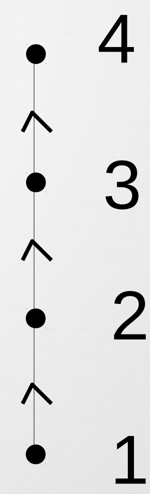



partial ordering is reflexive, remove all loops




Remove all edges that must be in the partial ordering because of transitivity



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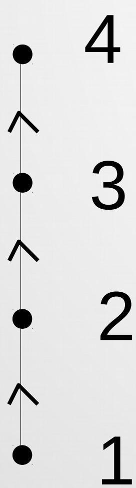
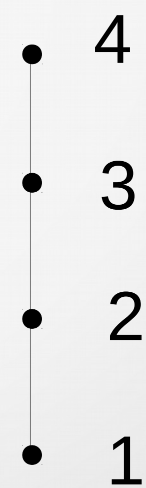



arrange each edge so that its initial vertex is below its terminal vertex, remove all the arrows – they all point upward

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**Example**: Draw the Hasse diagram representing the partial ordering { (a,b) | a divides b} on the set {1,2,3,5,10,11,15}

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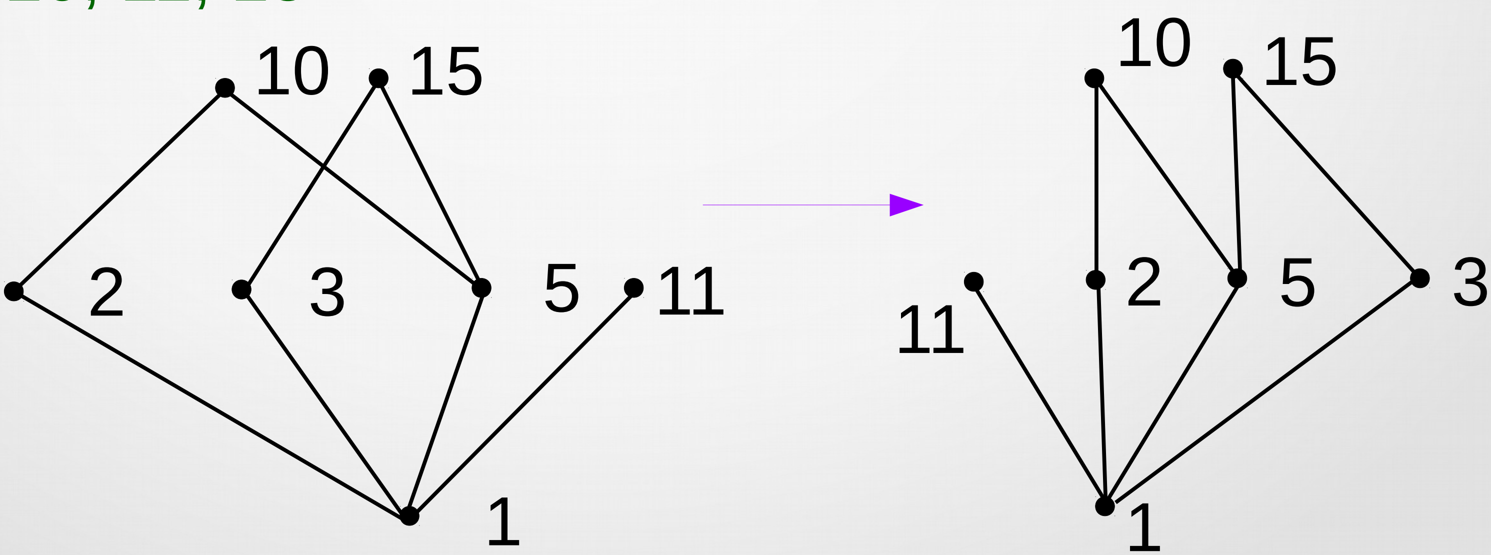
3 | 15
5 | 10, 15



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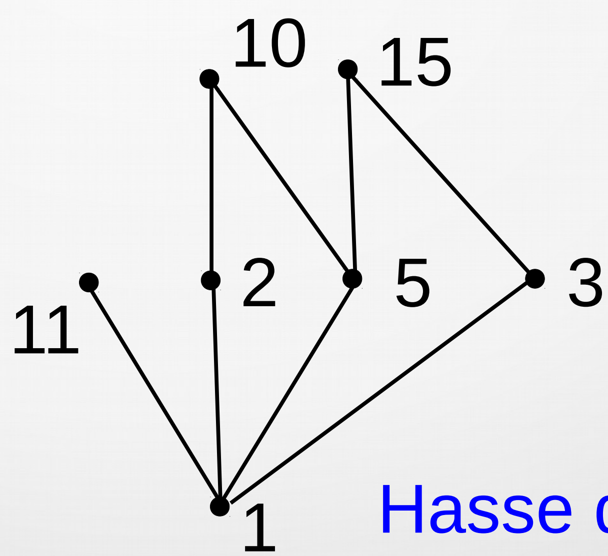


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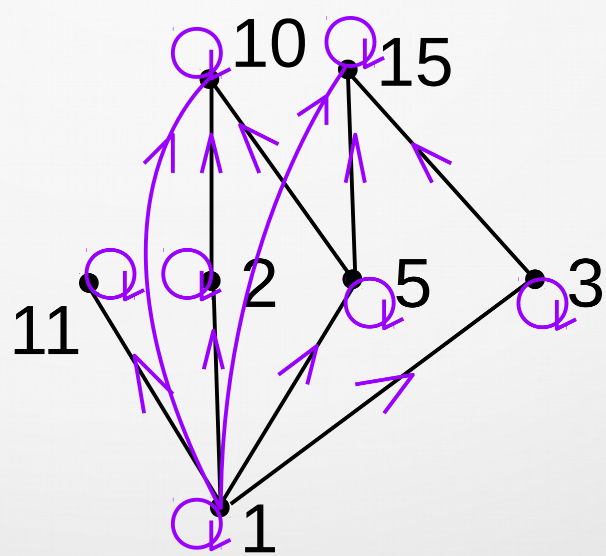

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## Hasse Diagrams

Let $(S, \preccurlyeq)$ be a *poset*. We say that element $y \in S$ *covers* an element $x \in S$ if $x \prec y$ and there is no element $z \in S$ such that $x \prec z \prec y$.

The set of pairs $(x,y)$ such that $y$ covers $x$ is called the *covering relation* of $(S, \preccurlyeq)$.

We can recover a poset from its covering relation.

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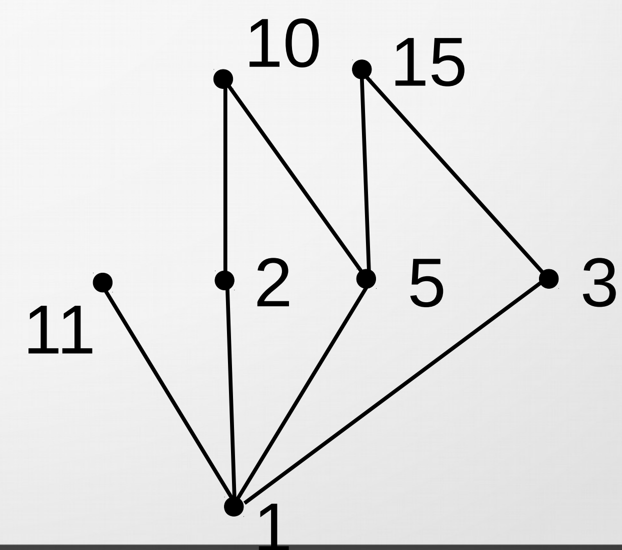
(1,11), (1,2), (1,5), (1,3), (2,10), (5,15), (3,15)



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a is the greatest element of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in S$. The greatest element is unique if it exists.

a is the least element of the poset $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$. The least element is unique if it exists.

# Section 9.6 *Partial orderings*

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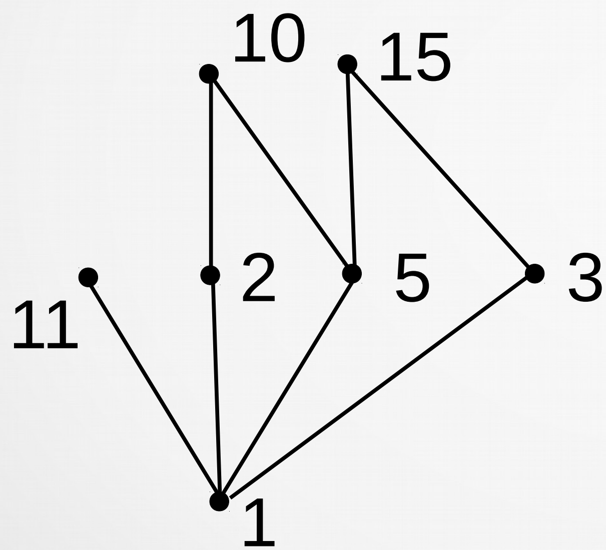


11,10, and 15 are *maximal elements*

1 is the *minimal element*

1 is the *least element*

No *greatest element*

## Section 9.6 *Partial orderings*

**Hasse Diagrams**

**[Def]** If u is an element of S such that a $\preccurlyeq$ u for all elements a ∊ A, then u is called the *upper bound* of A.

If l is an element of S such that l $\preccurlyeq$ a for all elements a ∊ A, then l is called the *lower bound* of A.

**Example**: find lower and upper bounds of the subsets {}, {}, and {} in the poset with the Hasse diagram

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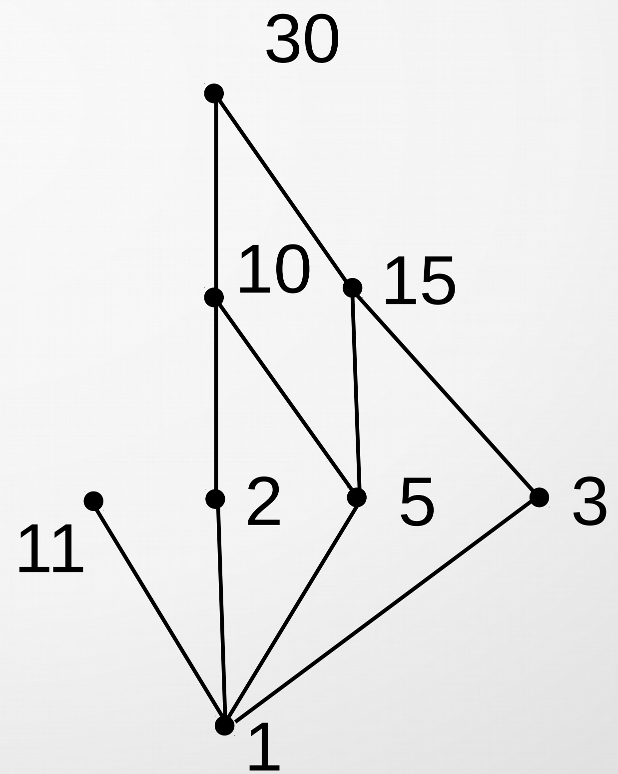
**Example**: find *lower* and *upper bounds* of the subsets {1, 2, 5}, {1, 2, 11}, and {10,15} in the poset with the Hasse diagram



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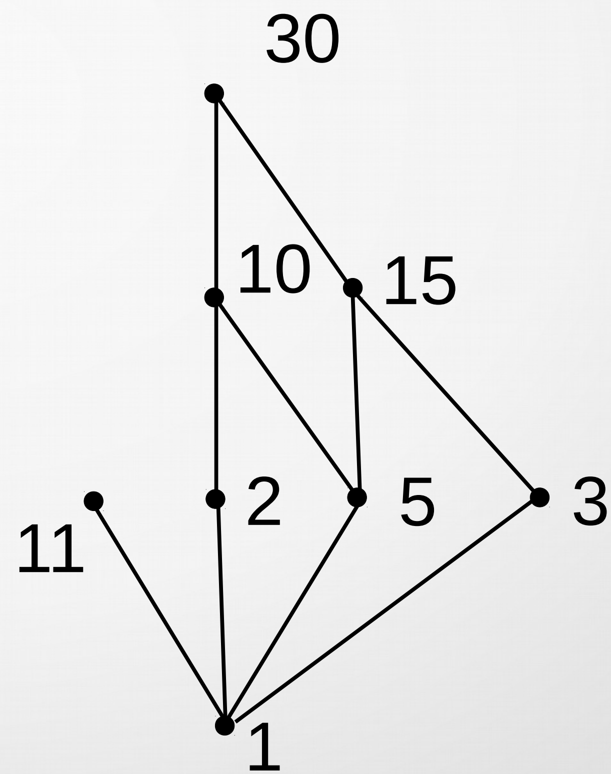


<u>Solution</u>:

Lower bound of {1,2,5}:

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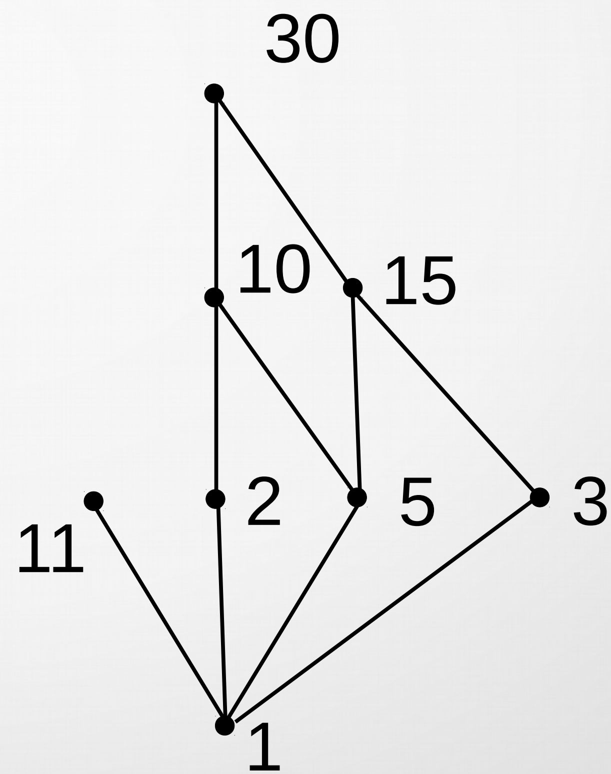


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Upper bounds of {1,2,5}: 10, 30

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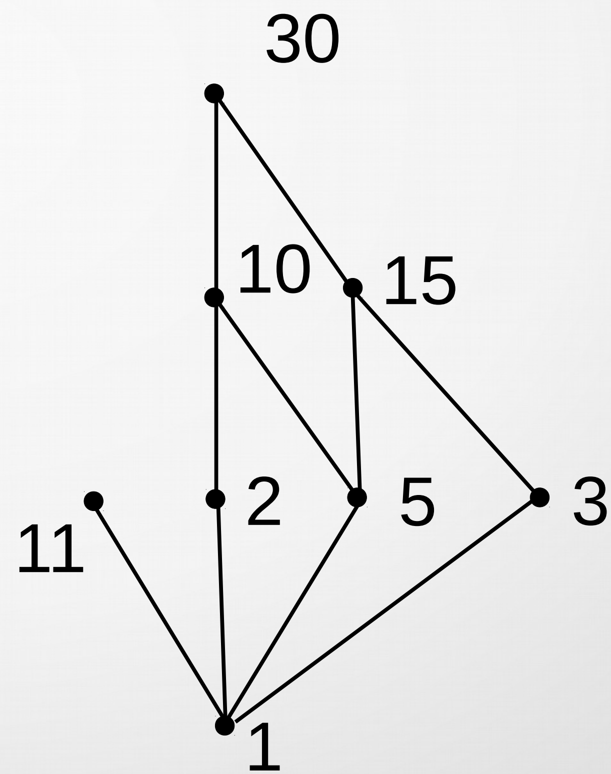


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