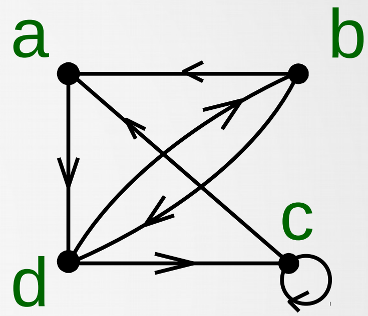
Section 9.3 *Representing relations*

Relations can be also represented with *directed graphs* (*digraphs*)

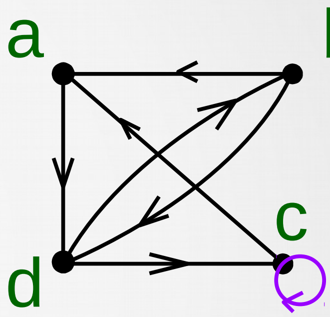
[Def] a directed graph (digraph) consists of:
- a set **V** of *vertices* (*nodes*)
- a set **E** of ordered pairs of elements of **V** called *edges* (*arcs*)



For any edge (*a*,*b*), vertex *a* is called the *initial vertex*, and vertex *b* called the *terminal vertex* of the edge.

Section 9.3 *Representing relations*

Relations can be also represented with *directed graphs* (*digraphs*)



[Def] a directed graph (digraph) consists of:
- a set **V** of *vertices* (*nodes*)
- a set **E** of ordered pairs of elements of **V** called *edges* (*arcs*)

For any edge (*a,b*), vertex *a* is called the *initial vertex*, and vertex *b* called the *terminal vertex* of the edge.

An edge (a,a) is represented using an arc from vertex a back to itself. Such an edge is called a *loop*.

Section 9.3 *Representing relations*

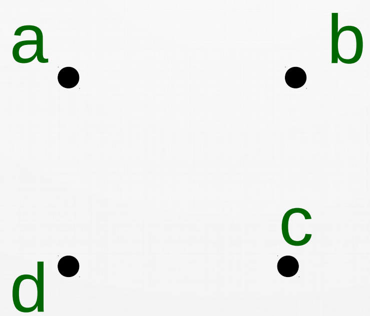
Example: See the digraph of the relation R on the set {a,b,c,d}

R = { (a,d), (b,a), (b,d), (c,a),(c,c), (d,b), (d,c) }

Section 9.3 *Representing relations*

Example: See the digraph of the relation *R* on the set {a,b,c,d}

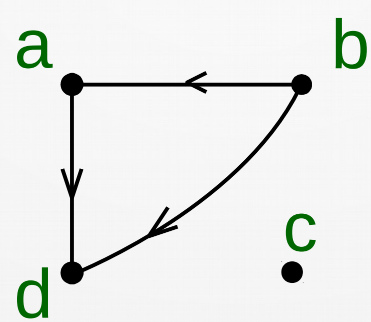
R = { (a,d), (b,a), (b,d), (c,a),(c,c), (d,b), (d,c) }



Section 9.3 *Representing relations*

Example: See the digraph of the relation *R* on the set {a,b,c,d}

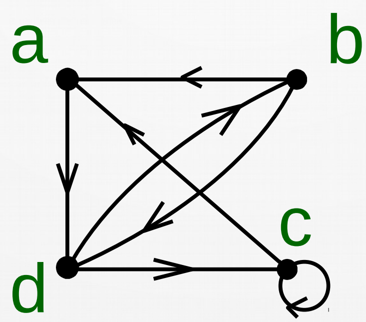
R = { (a,d), (b,a), (b,d), (c,a),(c,c), (d,b), (d,c) }



Section 9.3 *Representing relations*

Example: See the digraph of the relation *R* on the set {a,b,c,d}

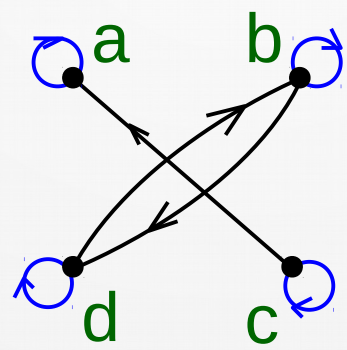
R = { (a,d), (b,a), (b,d), (c,a),(c,c), (d,b), (d,c) }



Section 9.3 *Representing relations*

Example: See the digraph of the relation *S* on the set {a,b,c,d}

S = { (a,a), (b,b), (b,d), (c,a),(c,c), (d,b), (d,d) }

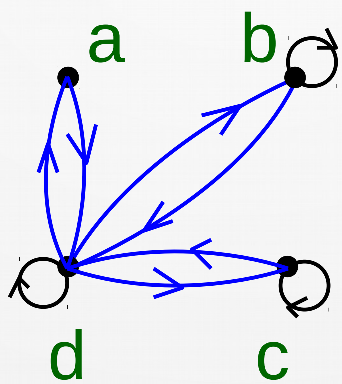


relation *S* is *reflexive*

Section 9.3 *Representing relations*

Example: See the digraph of the relation *T* on the set {a,b,c,d}

T = { (a,b), (b,b), (b,d), (c,c), (c,d), (d,a), (d,b), (d,c), (d,d) }



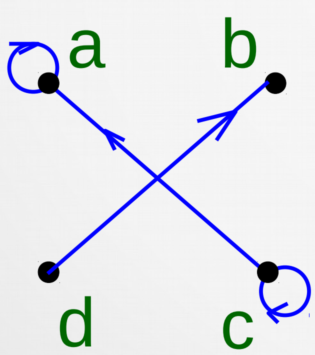
relation *T* is *symmetric*

Section 9.3 *Representing relations*

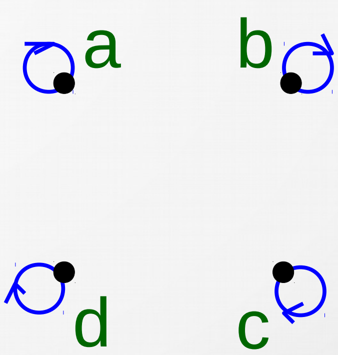
Example: See the digraph of the relations R_1 and R_2 on the set {a,b,c,d}

R_1 = { (a,a), (c,a),(c,c), (d,b) }

R_2 = { (a,a), (b,b), (c,c), (d,d) }



relation *R1* is *antisymmetric*

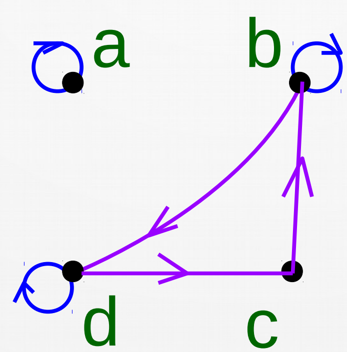


relation R_2 is *symmetric* and *antisymmetric*

Section 9.3 *Representing relations*

Example: See the digraph of the relation *T* on the set {a,b,c,d}

T = { (a,a), (b,b), (b,d), (c,b), (d,c), (d,d) }



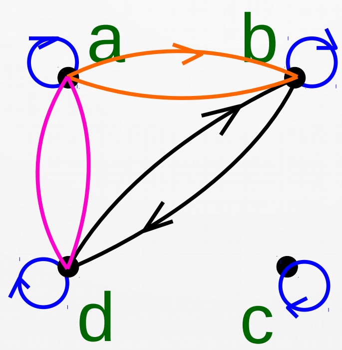
relation *T* is *transitive*

Section 9.3 *Representing relations*

We will talk more about graphs in Chapter 10

Section 9.5 *Equivalence relations*

[Def] A relation *R* on set A is called an *equivalence relation* if it is *reflexive*, *symmetric* and *transitive*.



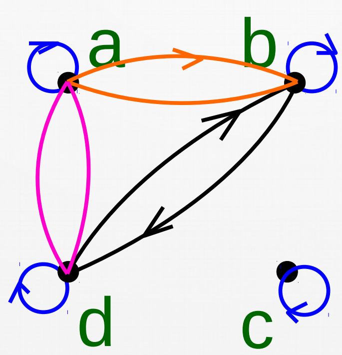
relation *R* is *reflexive, symmetric* and *transitive*

Section 9.5 *Equivalence relations*

[Def] A relation *R* on set A is called an *equivalence relation* if it is *reflexive*, *symmetric* and *transitive*.

[Def] Two elements a and b that are related by an equivalence relation are called *equivalent*.

denotation: a ~ b



relation *R* is *reflexive, symmetric* and *transitive*

Section 9.5 *Equivalence relations*

Example: congruence modulo m

Let $m \in Z^+$. The relation $R = \{(a,b) \mid a \equiv b\ (\textbf{mod}\ m)\}$ is an *equivalence relation*. Let's prove it.

Section 9.5 *Equivalence relations*

Example: congruence modulo *m*

Let $m \in Z^+$.

The *relation* $R = \{(a,b) \mid a \in Z, b \in Z, \text{ and } a \equiv b \ (\mathbf{mod}\ m)\}$ is an *equivalence relation*. Let's prove it.

To prove it we need to show that relation *R* is *reflexive, symmetric* and *transitive*

Recall the definition of congruence (*we will use it in the proof*):

$$a \equiv b \ (\mathbf{mod}\ m) \text{ iff } m \mid (a-b).$$

We also had a theorem:

$$a \equiv b \ (\mathbf{mod}\ m) \text{ iff } a(\mathbf{mod}\ m) = b(\mathbf{mod}\ m)$$

- we will use definition

Section 9.5 *Equivalence relations*

Example: congruence modulo *m*

Let $m \in Z^+$.

The relation $R = \{(a,b) \mid a \in Z, b \in Z, \text{ and } a \equiv b \ (\textbf{mod } m)\}$ is an *equivalence relation*. Let's prove it.

<u>Proof</u>:

1) *reflexivity*: we need to show that all $(a,a) \in R$

Section 9.5 *Equivalence relations*

Example: congruence modulo *m*

Let $m \in Z^+$.

The relation $R = \{(a,b) \mid a \in Z, b \in Z, \text{ and } a \equiv b\ (\textbf{mod}\ m)\}$ is an *equivalence relation*. Let's prove it.

Proof:

1) *reflexivity*:

$\forall a \in Z\ \ (a,a) \in R$, because $m \mid (a-a)$ i.e. $m \mid 0$, for any $m \in Z^+$.

Section 9.5 *Equivalence relations*

Example: congruence modulo ***m***

Let $m \in Z^+$.

The relation $R = \{(a,b) \mid a \in Z, b \in Z, \textit{and } a \equiv b \ (\textbf{mod } m)\}$ is an *equivalence relation*. Let's prove it.

<u>Proof</u>:

2) *symmetric*:

Assume $a \equiv b \ (\textbf{mod } m)$

Section 9.5 *Equivalence relations*

Example: congruence modulo m

Let $m \in Z^+$.

The relation $R = \{(a,b) \mid a \in Z, b \in Z, \text{ and } a \equiv b \ (\mathbf{mod}\ m)\}$ is an *equivalence relation*. Let's prove it.

Proof:

2) *symmetric*:

Assume $a \equiv b \ (\mathbf{mod}\ m)$

we need to show that in this case $b \equiv a \ (\mathbf{mod}\ m)$

Example: congruence modulo *m*

Let $m \in Z^+$.

The relation $R = \{(a,b) \mid a \in Z, b \in Z, \text{ and } a \equiv b \ (\textbf{mod}\ m)\}$ is an *equivalence relation*. Let's prove it.

Proof:

2) *symmetric*:

Assume $a \equiv b\ (\textbf{mod}\ m)$, then $m \mid (a-b)$, by def., i.e. $\exists k \in Z$ such that $a-b = km$.

Section 9.5 *Equivalence relations*

Example: congruence modulo m

Let $m \in Z^+$.

The relation $R = \{(a,b) \mid a \in Z, b \in Z, \text{ and } a \equiv b \ (\textbf{mod } m)\}$ is an *equivalence relation*. Let's prove it.

Proof:

2) *symmetric*:

Assume $a \equiv b \ (\textbf{mod } m)$, then $m \mid (a\text{-}b)$, by def., i.e. $\exists k \in Z$ such that $a\text{-}b = km$. In this case, $b - a = (-k)m$, which means that $m \mid (b\text{-}a)$, i.e. $b \equiv a \ (\textbf{mod } m)$

Section 9.5 *Equivalence relations*

Example: congruence modulo *m*

Let $m \in Z^+$.

The relation $R = \{(a,b) \mid a \in Z, b \in Z, \text{ and } a \equiv b \pmod{m}\}$ is an *equivalence relation*. Let's prove it.

Proof:

2) *symmetric*:

Assume $a \equiv b \pmod{m}$, then $m \mid (a-b)$, by def., i.e. $\exists k \in Z$ such that $a-b = km$. In this case, $b - a = (-k)m$, which means that $m \mid (b-a)$, i.e. $b \equiv a \pmod{m}$

We showed that congruence modulo is *symmetric*, i.e. if $(a,b) \in R$, then $(b,a) \in R$ as well.

Section 9.5 *Equivalence relations*

Example: congruence modulo m

Let $m \in Z^+$.

The relation $R = \{(a,b) \mid a \in Z, b \in Z, \text{ and } a \equiv b \ (\textbf{mod } m)\}$ is an *equivalence relation*. Let's prove it.

Proof:

3) *transitive*:

Assume $a \equiv b \ (\textbf{mod } m)$ and $b \equiv c \ (\textbf{mod } m)$

Section 9.5 *Equivalence relations*

Example: congruence modulo m

Let $m \in Z^{+}$.

The relation $R = \{(a,b) \mid a \in Z, b \in Z, \text{ and } a \equiv b \ (\textbf{mod } m)\}$ is an *equivalence relation*. Let's prove it.

Proof:

3) *transitive*:

Assume $a \equiv b \ (\textbf{mod } m)$ and $b \equiv c \ (\textbf{mod } m)$

we need to show that in this case $a \equiv c \ (\textbf{mod } m)$

Section 9.5 *Equivalence relations*

Example: congruence modulo m

Let $m \in Z^+$.

The relation $R = \{(a,b) \mid a \in Z, b \in Z, \text{ and } a \equiv b \ (\textbf{mod } m)\}$ is an *equivalence relation*. Let's prove it.

Proof:

3) *transitive*:

Assume $a \equiv b \ (\textbf{mod } m)$ and $b \equiv c \ (\textbf{mod } m)$, then

$m \mid (a-b)$, by def., i.e. $\exists k_1 \in Z$ such that $a-b = k_1 m$.

and

$m \mid (b-c)$, by def., i.e. $\exists k_2 \in Z$ such that $b-c = k_2 m$.

Section 9.5 *Equivalence relations*

Example: congruence modulo m

Let $m \in Z^+$.

The relation $R = \{(a,b) \mid a \in Z, b \in Z, \text{ and } a \equiv b \ (\textbf{mod } m)\}$ is an *equivalence relation*. Let's prove it.

Proof:

3) *transitive*:

Assume $a \equiv b$ (**mod** m) and $b \equiv c$ (**mod** m), then

m | (a-b), by def., i.e. $\exists k_1 \in Z$ such that $a\text{-}b = k_1 m.$

and $+$

m | (b-c), by def., i.e. $\exists k_2 \in Z$ such that $b\text{-}c = k_2 m.$

$a - c = m(k_1+k_2)$

Section 9.5 *Equivalence relations*

Example: congruence modulo *m*

Let $m \in Z^+$.

The relation $R = \{(a,b) \mid a \in Z, b \in Z, \text{ and } a \equiv b \ (\textbf{mod } m)\}$ is an *equivalence relation*. Let's prove it.

Proof:

3) *transitive*:

Assume $a \equiv b$ (**mod** m) and $b \equiv c$ (**mod** m), then

m | (a-b), by def., i.e. $\exists k_1 \in Z$ such that $a\text{-}b = k_1 m.$

and $+$

m | (b-c), by def., i.e. $\exists k_2 \in Z$ such that $b\text{-}c = k_2 m.$

$a - c = m(k_1+k_2)$, *therefore* $m \mid (a\text{-}c)$, *i.e.* $a \equiv c$ (**mod** m)

Section 9.5 *Equivalence relations*

Example: congruence modulo *m*

Let $m \in Z^+$.

The relation $R = \{(a,b) \mid a \in Z, b \in Z, \text{ and } a \equiv b \pmod{m}\}$ is an *equivalence relation*. Let's prove it.

Proof:

3) *transitive*:

Assume $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then

m | (a-b), by def., i.e. $\exists k_1 \in Z$ such that $a-b = k_1 m$.

and +

m | (b-c), by def., i.e. $\exists k_2 \in Z$ such that $b-c = k_2 m$.

$a - c = m(k_1+k_2)$, *therefore* $m \mid (a-c)$, *i.e.* $a \equiv c \pmod{m}$

We showed that if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$.

Section 9.5 *Equivalence relations*

Example: congruence modulo *m*

Let $m \in Z^+$.

The relation $R = \{(a,b) \mid a \in Z, b \in Z, \text{ and } a \equiv b \ (\textbf{mod}\ m)\}$ is an *equivalence relation*. Let's prove it.

Proof:

4) we showed that relation *R* is *reflexive*, *symmetric* and *transitive*, therefore it is an *equivalence* relation.

q.e.d.

Section 9.5 *Equivalence relations*

Equivalence classes

[Def] Let *R* be an equivalence relation on a set A. The set of all elements that are related to an element *a* of A is called the *equivalence class* of *a*.

denotation: $[a]_R$ equivalence class of *a*

when only one relation is under consideration: $[a]$

Section 9.5 *Equivalence relations*

Equivalence classes

[Def] Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a.

denotation: $[a]_R$ equivalence class of a

when only one relation is under consideration: $[a]$

In other words, if R is an equivalence relation on a set A, the equivalence class of the element a is

$[a]_R = \{s \mid (a,s) \in R\}$

Section 9.5 *Equivalence relations*

Equivalence classes

[Def] Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a.

denotation: $[a]_R$ equivalence class of a

when only one relation is under consideration: $[a]$

In other words, if R is an equivalence relation on a set A, the equivalence class of the element a is

$[a]_R = \{s \mid (a,s) \in R\}$ Note that $[a]_R \neq \emptyset$ because R is reflexive

Section 9.5 *Equivalence relations*

Equivalence classes

[Def] Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a.

denotation: $[a]_R$ equivalence class of a

when only one relation is under consideration: $[a]$

In other words, if R is an equivalence relation on a set A, the equivalence class of the element a is

$[a]_R = \{s \mid (a,s) \in R\}$

If $b \in [a]_R$, then b is called *representative* of this equivalence class.

Section 9.5 *Equivalence relations*

Example: What are the equivalence classes of *0* and *1* for *congruence* **modulo** 7?

Section 9.5 *Equivalence relations*

Example: What are the equivalence classes of *0* and *1* for *congruence* **modulo** 7?

Solution:

Let's find the equivalence class of 0:

Section 9.5 *Equivalence relations*

Example: What are the equivalence classes of *0* and *1* for *congruence* **modulo** 7?

Solution:

Let's find the equivalence class of 0:

0 **mod** *7* = *0*, *7* **mod** *7* = *0*, *14* **mod** *7* = *0*, *-7* **mod** *7* = *0*…

Section 9.5 *Equivalence relations*

Example: What are the equivalence classes of *0* and *1* for *congruence* **modulo** 7?

Solution:

Let's find the equivalence class of 0:

0 **mod** *7* = *0*, *7* **mod** *7* = *0*, *14* **mod** *7* = *0*, *-7* **mod** *7* = *0*…

Therefore, $[0] = \{\ldots, -14, -7, 0, 7, 14, 21, \ldots\}$ or

$[0] = \{ a \mid \exists k \in \mathbf{Z} \text{ and } a = 7k\}$.

Section 9.5 *Equivalence relations*

Example: What are the equivalence classes of *0* and *1* for *congruence* **modulo** 7?

Solution:

Let's find the equivalence class of 0:

0 **mod** *7* = *0*, *7* **mod** *7* = *0*, *14* **mod** *7* = *0*, *-7* **mod** *7* = *0*…

Therefore, $[0] = \{\ldots, -14, -7, 0, 7, 14, 21, \ldots\}$ or

$[0] = \{ a \mid \exists k \in \mathbf{Z} \text{ and } a = 7k\}$.

Let's find the equivalence class of 1:

Section 9.5 *Equivalence relations*

Example: What are the equivalence classes of *0* and *1* for *congruence* **modulo** 7?

<u>Solution</u>:

Let's find the equivalence class of 0:

0 **mod** *7* = *0*, *7* **mod** *7* = *0*, *14* **mod** *7* = *0*, *-7* **mod** *7* = *0*…

Therefore, [0] = {*…, -14, -7, 0, 7, 14, 21,...*} or

[0] = { *a* | ∃*k* ∈ **Z** and *a* = *7k*}.

Let's find the equivalence class of 1:

1 **mod** *7* = *1*, *8* **mod** *7* = *1*, *15* **mod** *7* = *1*, *-6* **mod** *7* = *1*…

Section 9.5 *Equivalence relations*

Example: What are the equivalence classes of *0* and *1* for *congruence* **modulo** 7?

Solution:

Let's find the equivalence class of 0:

0 **mod** *7* = *0*, *7* **mod** *7* = *0*, *14* **mod** *7* = *0*, *-7* **mod** *7* = *0*…

Therefore, [0] = {*…, -14, -7, 0, 7, 14, 21,...*} or

[0] = { *a* | ∃*k* ∈ **Z** and *a* = *7k*}.

Let's find the equivalence class of 1:

1 **mod** *7* = *1*, *8* **mod** *7* = *1*, *15* **mod** *7* = *1*, *-6* **mod** *7* = *1*…

Therefore, [1] = {*…, -13, -6, 0, 8, 15, 22,...*} or

[1] = { *a* | ∃*k* ∈ **Z** and *a* = *7k+1*}.

Section 9.5 *Equivalence relations*

Example: What are the equivalence classes of *0* and *1* for *congruence* **modulo** 7?

<u>**Solution**</u>**:**

Let's find the equivalence class of 0:

0 **mod** *7* = *0*, *7* **mod** *7* = *0*, *14* **mod** *7* = *0*, *-7* **mod** *7* = *0*…

Therefore, [0] = {*…, -14, -7, 0, 7, 14, 21,...*} or
[0] = { *a* | $\exists k \in$ **Z** and *a* = *7k*}.

Let's find the equivalence class of 1:

1 **mod** *7* = *1*, *8* **mod** *7* = *1*, *15* **mod** *7* = *1*, *-6* **mod** *7* = *1*…

Therefore, [1] = {*…, -13, -6, 0, 8, 15, 22,...*} or
[1] = { *a* | $\exists k \in$ **Z** and *a* = *7k+1*}.

Section 9.5 *Equivalence relations*

Example: Let *S* be the relation on the set of all sets of real numbers such that *A S B* iff *A* and *B* have the same cardinality.

1) Is *S* an equivalence relation?
2) What is the equivalence class of the set {0,1,2} ?
3) What is the equivalence class of the set **Z**?

Section 9.5 *Equivalence relations*

Example: Let *S* be the relation on the set of all sets of real numbers such that *A S B* iff *A* and *B* have the same cardinality.

1) Is *S* an equivalence relation?
2) What is the equivalence class of the set {0,1,2} ?
3) What is the equivalence class of the set **Z**?

Notes:

a) elements are **sets**, not just numbers, or strings, …

b)Set of all sets of real numbers: powerset of ***R**, set of all real numbers*

Section 9.5 *Equivalence relations*

Example: Let *S* be the relation on the set of all sets of real numbers such that *A S B* iff *A* and *B* have the same cardinality.

1) Is *S* an equivalence relation?
2) What is the equivalence class of the set {0,1,2} ?
3) What is the equivalence class of the set **Z**?

Solution:

1) is relation *S* *reflexive*? A *S* A for any A ∈ P(**R**)?

Section 9.5 *Equivalence relations*

Example: Let *S* be the relation on the set of all sets of real numbers such that *A S B* iff *A* and *B* have the same cardinality.

1) Is *S* an equivalence relation?
2) What is the equivalence class of the set {0,1,2} ?
3) What is the equivalence class of the set **Z**?

Solution:

1) is relation *S* *reflexive*? A *S* A for any A $\in$ P(**R**)?

Yes, because |A| = |A| it is *reflexive*

Section 9.5 *Equivalence relations*

Example: Let S be the relation on the set of all sets of real numbers such that $A\ S\ B$ iff A and B have the same cardinality.

1) Is S an equivalence relation?
2) What is the equivalence class of the set {0,1,2} ?
3) What is the equivalence class of the set **Z**?

Solution:

1) is relation S *reflexive*? A S A for any $A \in P(\mathbf{R})$?

Yes, because |A| = |A| it is *reflexive*

is relation S *symmetric*? If A S B then B S A for $A,B \in P(\mathbf{R})$?

Section 9.5 *Equivalence relations*

Example: Let *S* be the relation on the set of all sets of real numbers such that *A S B* iff *A* and *B* have the same cardinality.

1) Is *S* an equivalence relation?
2) What is the equivalence class of the set {0,1,2} ?
3) What is the equivalence class of the set **Z**?

Solution:

1) is relation *S reflexive*? A *S* A for any A ∈ P(**R**)?

Yes, because |A| = |A| it is *reflexive*

is relation *S symmetric*? If A *S* B then B *S* A for A,B∈ P(**R**)?

Yes, because |A| = |B| → |B| = |A| it is *symmetric*

Section 9.5 *Equivalence relations*

Example: Let *S* be the relation on the set of all sets of real numbers such that *A S B* iff *A* and *B* have the same cardinality.

1) Is *S* an equivalence relation?
2) What is the equivalence class of the set {0,1,2} ?
3) What is the equivalence class of the set **Z**?

Solution:

1) is relation *S* *reflexive*? A *S* A for any A $\in$ P(**R**)?
Yes, because |A| = |A| it is *reflexive*
is relation *S* *symmetric*? If A *S* B then B *S* A for A,B$\in$ P(**R**)?
Yes, because |A| = |B| $\rightarrow$ |B| = |A| it is *symmetric*
is relation *S* *transitive*? If A *S* B and B *S* C then A *S* C?

Section 9.5 *Equivalence relations*

Example: Let *S* be the relation on the set of all sets of real numbers such that *A S B* iff *A* and *B* have the same cardinality.

1) Is *S* an equivalence relation?
2) What is the equivalence class of the set {0,1,2} ?
3) What is the equivalence class of the set **Z**?

Solution:

1) is relation *S reflexive*? A *S* A for any A $\in$ P(**R**)?
Yes, because |A| = |A| it is *reflexive*

is relation *S symmetric*? If A *S* B then B *S* A for A,B$\in$ P(**R**)?
Yes, because |A| = |B| $\rightarrow$ |B| = |A| it is *symmetric*

is relation *S transitive*? If A *S* B and B *S* C then A *S* C?
Yes, because if |A| = |B| and |B| = |C|, then |A|=|C| it is *transitive.*

Section 9.5 *Equivalence relations*

Example: Let *S* be the relation on the set of all sets of real numbers such that *A S B* iff *A* and *B* have the same cardinality.

1) Is *S* an equivalence relation?
2) What is the equivalence class of the set {0,1,2} ?
3) What is the equivalence class of the set **Z**?

Solution:

1) is relation *S* *reflexive*? A *S* A for any A ∈ P(**R**)?

Yes, because |A| = |A| it is *reflexive*

is relation *S* *symmetric*? If A *S* B then B *S* A for A,B∈ P(**R**)?

Yes, because |A| = |B| → |B| = |A| it is *symmetric*

is relation *S* *transitive*? If A *S* B and B *S* C then A *S* C?

Yes, because if |A| = |B| and |B| = |C|, then |A|=|C| it is *transitive.* Therefore it is an *equivalence* relation.

Section 9.5 *Equivalence relations*

Example: Let *S* be the relation on the set of all sets of real numbers such that *A S B* iff *A* and *B* have the same cardinality.

1) Is *S* an equivalence relation?
2) What is the equivalence class of the set {0,1,2} ?
3) What is the equivalence class of the set **Z**?

Solution:

2)

Section 9.5 *Equivalence relations*

Example: Let *S* be the relation on the set of all sets of real numbers such that *A S B* iff *A* and *B* have the same cardinality.

1) Is *S* an equivalence relation?
2) What is the equivalence class of the set {0,1,2} ?
3) What is the equivalence class of the set **Z**?

Solution:

2) set {0,1,2} has three elements, i.e. |{1,2,3}| = 3,

Section 9.5 *Equivalence relations*

Example: Let *S* be the relation on the set of all sets of real numbers such that *A S B* iff *A* and *B* have the same cardinality.

1) Is *S* an equivalence relation?
2) What is the equivalence class of the set {0,1,2} ?
3) What is the equivalence class of the set **Z**?

Solution:

2) set {0,1,2} has three elements, i.e. |{1,2,3}| = 3,
Therefore any three-element set is related to the set {1,2,3}

Section 9.5 *Equivalence relations*

Example: Let *S* be the relation on the set of all sets of real numbers such that *A S B* iff *A* and *B* have the same cardinality.

1) Is *S* an equivalence relation?
2) What is the equivalence class of the set {0,1,2} ?
3) What is the equivalence class of the set **Z**?

Solution:

2) set {0,1,2} has three elements, i.e. $|\{1,2,3\}| = 3$,
Therefore any three-element set is related to the set {1,2,3}

$[\{1,2,3\}] = \{A \mid |A| = 3 \text{ and elements of } A \text{ are real numbers}\}$

or

$[\{1,2,3\}] = \{A \mid |A| = 3 \text{ and } A,B \in P(\mathbf{R})\}$

Section 9.5 *Equivalence relations*

Example: Let *S* be the relation on the set of all sets of real numbers such that *A S B* iff *A* and *B* have the same cardinality.

1) Is *S* an equivalence relation?
2) What is the equivalence class of the set {0,1,2} ?
3) What is the equivalence class of the set **Z**?

Solution:

3)

Section 9.5 *Equivalence relations*

Example: Let *S* be the relation on the set of all sets of real numbers such that *A S B* iff *A* and *B* have the same cardinality.

1) Is *S* an equivalence relation?
2) What is the equivalence class of the set {0,1,2} ?
3) What is the equivalence class of the set **Z**?

Solution:

3) set **Z** is infinite, i.e. $|\mathbf{Z}| = \infty$

Section 9.5 *Equivalence relations*

Example: Let *S* be the relation on the set of all sets of real numbers such that *A S B* iff *A* and *B* have the same cardinality.

1) Is *S* an equivalence relation?
2) What is the equivalence class of the set {0,1,2} ?
3) What is the equivalence class of the set **Z**?

Solution:

3) set **Z** is infinite, i.e. $|\mathbf{Z}| = \infty$, so any infinite set composed of real numbers is related to **Z**

Section 9.5 *Equivalence relations*

Example: Let *S* be the relation on the set of all sets of real numbers such that *A S B* iff *A* and *B* have the same cardinality.

1) Is *S* an equivalence relation?
2) What is the equivalence class of the set {0,1,2} ?
3) What is the equivalence class of the set **Z**?

Solution:

3) set **Z** is infinite, i.e. $|\mathbf{Z}| = \infty$, so any infinite set composed of real numbers is related to **Z**

$[Z] = \{A \mid |A| = \infty \text{ and } A \in P(\mathbf{R})\}$

Section 9.5 *Equivalence relations*

Equivalence classes and partitions

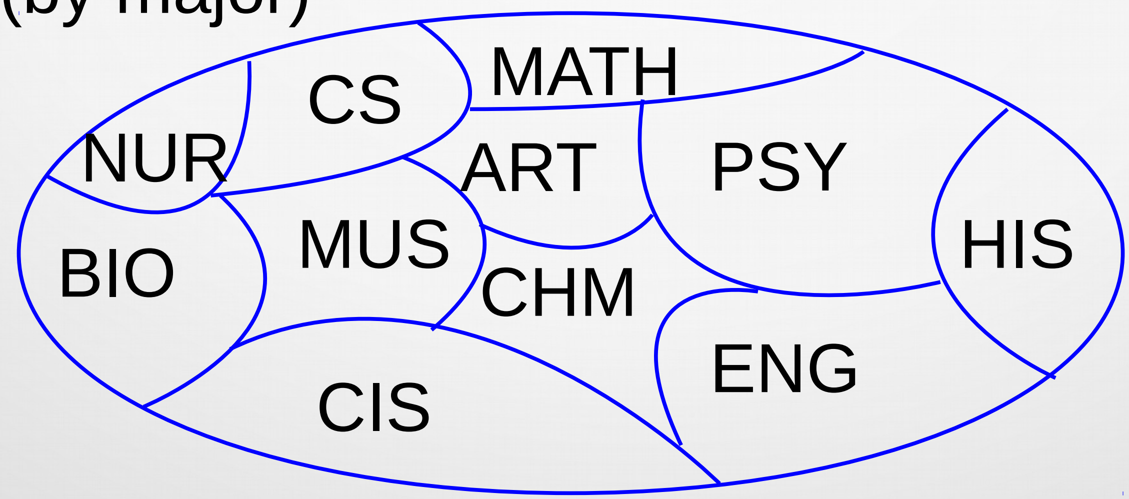
Let A be the set of students at BCC who are majoring in exactly on subject, and let R be the relation on A consisting of pairs (x,y), where a and y are students majoring in the same major.

Section 9.5 *Equivalence relations*

Equivalence classes and partitions

Let A be the set of students at BCC who are majoring in exactly on subject, and let *R* be the relation on A consisting of pairs *(x,y)*, where *a* and *y* are students majoring in the same major.

R splits all students in A into a collection of disjoint subsets (by major)

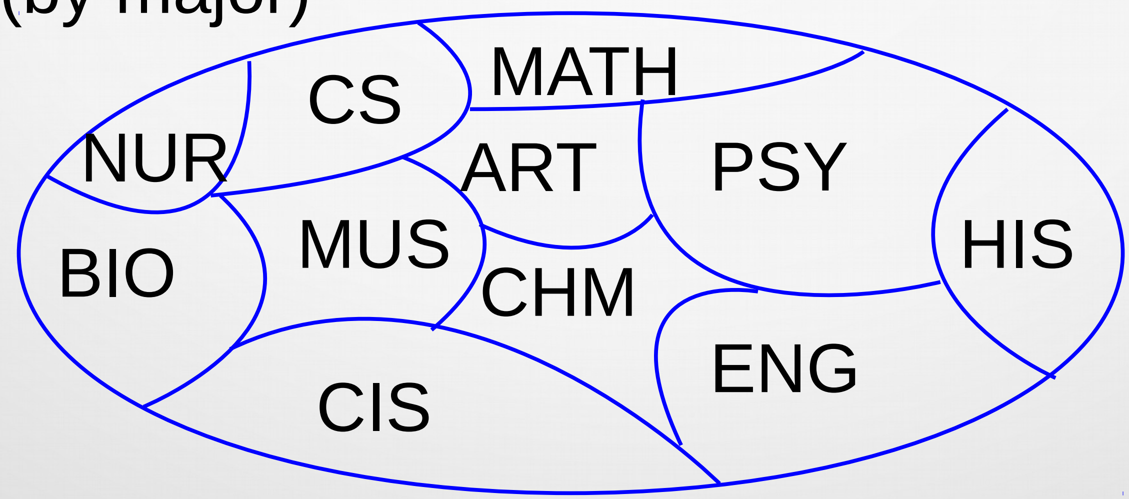


Section 9.5 *Equivalence relations*

Equivalence classes and partitions

Let A be the set of students at BCC who are majoring in exactly on subject, and let *R* be the relation on A consisting of pairs *(x,y)*, where *a* and *y* are students majoring in the same major.

R splits all students in A into a collection of disjoint subsets (by major)



partition of a set into disjoint non-empty subsets

Section 9.5 *Equivalence relations*

[Theorem] Let *R* be an equivalence relation on a set A. The following statements for elements *a* and *b* of A are equivalent:

(1) $a \, R \, b$

(2) $[a] = [b]$

(3) $[a] \cap [b] \neq \varnothing$

<u>**Proof**</u>:

1) let's show that (1) → (2)

2) let's show that (2) → (3)

3) let's show that (3) → (1)

This is enough to show that all three statements are equivalent.

Section 9.5 *Equivalence relations*

[Theorem] Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent:

(1) $a\ R\ b$ (2) $[a] = [b]$ (3) $[a] \cap [b] \neq \emptyset$

Proof:

1) let's show that (1) $\rightarrow$ (2): assume that $a\ R\ b$.

Let's show that in this case $[a] \subseteq [b]$ and $[b] \subseteq [a]$:

Section 9.5 *Equivalence relations*

[Theorem] Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent:

(1) $a \, R \, b$ (2) $[a] = [b]$ (3) $[a] \cap [b] \neq \emptyset$

Proof:

1) let's show that (1) $\rightarrow$ (2): assume that $a \, R \, b$.

Let's show that in this case $[a] \subseteq [b]$ and $[b] \subseteq [a]$:

suppose $c \in [a]$,

Section 9.5 *Equivalence relations*

[Theorem] Let *R* be an equivalence relation on a set A. The following statements for elements *a* and *b* of A are equivalent:

(1) $a\ R\ b$ (2) $[a] = [b]$ (3) $[a] \cap [b] \neq \emptyset$

Proof:

1) let's show that (1) $\rightarrow$ (2): assume that $a\ R\ b$.

Let's show that in this case $[a] \subseteq [b]$ and $[b] \subseteq [a]$:

suppose $c \in [a]$, then (by def of equiv. class) $a\ R\ c$, and $c\ R\ a$ (since *R* is symmetric).

Section 9.5 *Equivalence relations*

[Theorem] Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent:

(1) $a \, R \, b$ (2) $[a] = [b]$ (3) $[a] \cap [b] \neq \emptyset$

<u>Proof</u>:

1) let's show that (1) $\rightarrow$ (2): assume that $a \, R \, b$.

Let's show that in this case $[a] \subseteq [b]$ and $[b] \subseteq [a]$:

suppose $c \in [a]$, then (by def of equiv. class) $a \, R \, c$, and $c \, R \, a$ (since R is symmetric).

Combining $c \, R \, a$ and $a \, R \, b$ we get $c \, R \, b$ (since R *is transitive*),

Section 9.5 *Equivalence relations*

[Theorem] Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent:

(1) $a\ R\ b$ (2) $[a] = [b]$ (3) $[a] \cap [b] \neq \emptyset$

Proof:

1) let's show that (1) $\rightarrow$ (2): assume that $a\ R\ b$.

Let's show that in this case $[a] \subseteq [b]$ and $[b] \subseteq [a]$:

suppose $c \in [a]$, then (by def of equiv. class) $a\ R\ c$, and $c\ R\ a$ (since R is symmetric).

Combining $c\ R\ a$ and $a\ R\ b$ we get $c\ R\ b$ (since R *is transitive*),

followed by $b\ R\ c$ (since R is symmetric).

Section 9.5 *Equivalence relations*

[Theorem] Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent:

(1) $a R b$ (2) $[a] = [b]$ (3) $[a] \cap [b] \neq \varnothing$

Proof:

1) let's show that (1) $\rightarrow$ (2): assume that $a R b$.

Let's show that in this case $[a] \subseteq [b]$ and $[b] \subseteq [a]$:

suppose $c \in [a]$, then (by def of equiv. class) $a R c$, and $c R a$ (since R is symmetric).

Combining $c R a$ and $a R b$ we get $c R b$ (since R *is transitive)*,

followed by $b R c$ (since R is symmetric).

We showed that if $c \in [a]$, then $c \in [b]$, i.e. $[a] \subseteq [b]$

Section 9.5 *Equivalence relations*

[Theorem] Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent:

(1) $a \; R \; b$ (2) $[a] = [b]$ (3) $[a] \cap [b] \neq \varnothing$

<u>Proof</u>:

1) let's show that (1) $\rightarrow$ (2): assume that $a \; R \; b$.

Let's show that in this case $[a] \subseteq [b]$ and $[b] \subseteq [a]$:

suppose $c \in [a]$, then (by def of equiv. class) $a \; R \; c$, and $c \; R \; a$ (since R is symmetric).

Combining $c \; R \; a$ and $a \; R \; b$ we get $c \; R \; b$ (since R *is transitive*),

followed by $b \; R \; c$ (since R is symmetric).

We showed that if $c \in [a]$, then $c \in [b]$, i.e. $[a] \subseteq [b]$

The proof of $[b] \subseteq [a]$ is similar.

Section 9.5 *Equivalence relations*

[Theorem] Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent:

(1) $a\ R\ b$ (2) $[a] = [b]$ (3) $[a] \cap [b] \neq \emptyset$

Proof:

2) let's show that (2) $\rightarrow$ (3): assume that $[a] = [b]$.

Section 9.5 *Equivalence relations*

[Theorem] Let *R* be an equivalence relation on a set A. The following statements for elements *a* and *b* of A are equivalent:

(1) $a\ R\ b$ (2) $[a] = [b]$ (3) $[a] \cap [b] \neq \emptyset$

Proof:

2) let's show that (2) → (3): assume that $[a] = [b]$.
This means that every element from $[a]$ is an element of $[b]$ and vice versa

Section 9.5 *Equivalence relations*

[Theorem] Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent:

(1) $a \, R \, b$ (2) $[a] = [b]$ (3) $[a] \cap [b] \neq \emptyset$

Proof:

2) let's show that (2) $\rightarrow$ (3): assume that $[a] = [b]$.
This means that every element from $[a]$ is an element of $[b]$ and vice versa, so $[a] \cap [b] \neq \emptyset$

Section 9.5 *Equivalence relations*

[Theorem] Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent:

(1) $a R b$ (2) $[a] = [b]$ (3) $[a] \cap [b] \neq \varnothing$

Proof:

2) let's show that (2) $\rightarrow$ (3): assume that $[a] = [b]$.
This means that every element from $[a]$ is an element of $[b]$ and vice versa, so $[a] \cap [b] \neq \varnothing$

Note that $[a]$ and $[b]$: are not empty sets (*reflexivity* takes care of it)

Section 9.5 *Equivalence relations*

[Theorem] Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent:

(1) $a \, R \, b$ (2) $[a] = [b]$ (3) $[a] \cap [b] \neq \varnothing$

Proof:

3) let's show that (3) $\rightarrow$ (1):

Section 9.5 *Equivalence relations*

[Theorem] Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent:

(1) $a\ R\ b$ (2) $[a] = [b]$ (3) $[a] \cap [b] \neq \emptyset$

Proof:

3) let's show that (3) → (1): assume that $[a] \cap [b] \neq \emptyset$

Section 9.5 *Equivalence relations*

[Theorem] Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent:

(1) $a R b$ (2) $[a] = [b]$ (3) $[a] \cap [b] \neq \emptyset$

Proof:

3) let's show that (3) $\rightarrow$ (1): assume that $[a] \cap [b] \neq \emptyset$

It means that there exists at least one element $c \in [a]$ and $c \in [b]$

Section 9.5 *Equivalence relations*

[Theorem] Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent:

(1) $a \, R \, b$ (2) $[a] = [b]$ (3) $[a] \cap [b] \neq \emptyset$

Proof:

3) let's show that (3) $\rightarrow$ (1): assume that $[a] \cap [b] \neq \emptyset$

It means that there exists at least one element $c \in [a]$ and $c \in [b]$, hence $a \, R \, c$ and $b \, R \, c$.

Section 9.5 *Equivalence relations*

[Theorem] Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent:

(1) $a \; R \; b$ (2) $[a] = [b]$ (3) $[a] \cap [b] \neq \emptyset$

Proof:

3) let's show that (3) → (1): assume that $[a] \cap [b] \neq \emptyset$

It means that there exists at least one element $c \in [a]$ and $c \in [b]$, hence $a \; R \; c$ and $b \; R \; c$.

Since R is symmetric we get $c \; R \; b$

Section 9.5 *Equivalence relations*

[Theorem] Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent:

(1) $a\ R\ b$ (2) $[a] = [b]$ (3) $[a] \cap [b] \neq \varnothing$

Proof:

3) let's show that (3) $\rightarrow$ (1): assume that $[a] \cap [b] \neq \varnothing$

It means that there exists at least one element $c \in [a]$ and $c \in [b]$, hence $a\ R\ c$ and $b\ R\ c$.

Since R is symmetric we get $c\ R\ b$

Hence from $a\ R\ c$ and $c\ R\ b$ we get $a\ R\ b$ (since R is transitive). This completes 3)

Section 9.5 *Equivalence relations*

[Theorem] Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent:

(1) $a\ R\ b$ (2) $[a] = [b]$ (3) $[a] \cap [b] \neq \varnothing$

<u>**Proof**</u>:

3) let's show that (3) → (1): assume that $[a] \cap [b] \neq \varnothing$

It means that there exists at least one element $c \in [a]$ and $c \in [b]$, hence $a\ R\ c$ and $b\ R\ c$.

Since R is symmetric we get $c\ R\ b$

Hence from $a\ R\ c$ and $c\ R\ b$ we get $a\ R\ b$ (since R is transitive). This completes 3)

We proved that (1) → (2), (2) → (3), and (3) → (1).

It means that all three statements are equivalent.

q.e.d.

Section 9.5 *Equivalence relations*

Partitions

Let R be an equivalence relation on a set A. Then

$$\bigcup_{a \in A} [a]_R = A$$

i.e. the union of the equivalence classes of R is all A.

Section 9.5 *Equivalence relations*

Partitions

Let R be an equivalence relation on a set A. Then

$$\bigcup_{a \in A} [a]_R = A$$

i.e. the union of the equivalence classes of R is all A.

Moreover, $[a]_R \cap [b]_R = \emptyset$ if $[a]_R \neq [b]_R$

i.e. these equivalence classes are either equal or disjoint.

Section 9.5 *Equivalence relations*

Partitions

Let R be an equivalence relation on a set A. Then

$$\bigcup_{a \in A} [a]_R = A$$

i.e. the union of the equivalence classes of R is all A.

Moreover, $[a]_R \cap [b]_R = \varnothing$ if $[a]_R \neq [b]_R$

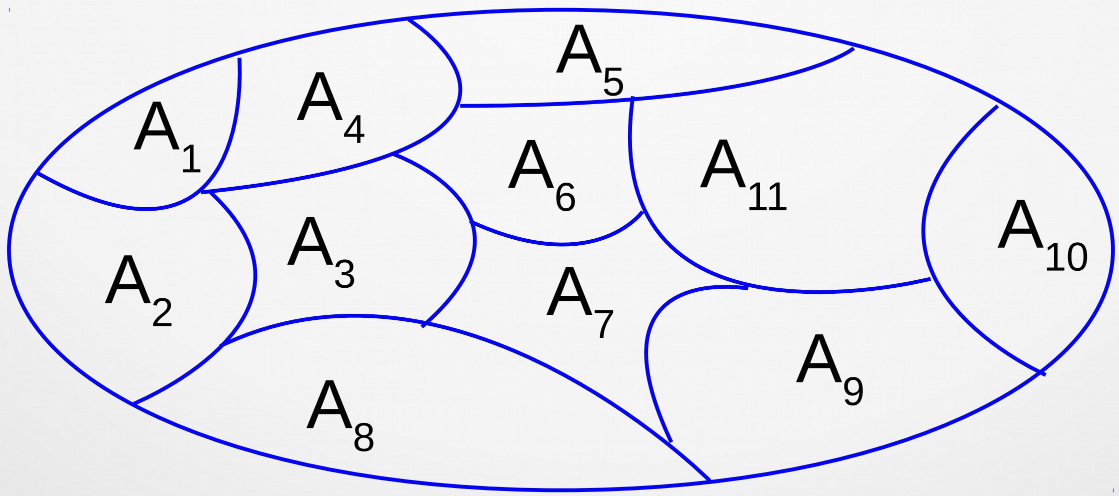
i.e. these equivalence classes are either equal or disjoint.

So equivalence classes form a *partition* of set A (disjoint subsets)

Partitions

[Def] A collection of subsets A_i , $i \in I$ (I is an index set) forms a *partition* of set S iff

$A_i \neq \emptyset \quad \forall i \in I$, $\quad A_i \cap A_j = \emptyset$ when $i \neq j$, and $\bigcup_{i \in I} A_i = S$



Section 9.5 *Equivalence relations*

Partitions

Example: Suppose that $S = \{a,b,c,d,e,f\}$. Which collections of sets form a partition of S?

a) $A_1 = \{a,b,c,d\}$, $A_2 = \{a,e\}$, $A_3 = \{f\}$

b) $A_1 = \{a,b\}$, $A_2 = \{c,d\}$, $A_3 = \{e,f\}$

c) $A_1 = \{a\}$, $A_2 = \{b,c,d\}$, $A_3 = \{e\}$

Section 9.5 *Equivalence relations*

Partitions

Example: Suppose that $S = \{a,b,c,d,e,f\}$. Which collections of sets form a partition of S?

a) $A_1 = \{a,b,c,d\}$, $A_2 = \{a,e\}$, $A_3 = \{f\}$

does not form a partition of S because $A_1 \cap A_2 \neq \emptyset$

b) $A_1 = \{a,b\}$, $A_2 = \{c,d\}$, $A_3 = \{e,f\}$

c) $A_1 = \{a\}$, $A_2 = \{b,c,d\}$, $A_3 = \{e\}$

Section 9.5 *Equivalence relations*

Partitions

Example: Suppose that $S = \{a,b,c,d,e,f\}$. Which collections of sets form a partition of S?

a) $A_1 = \{a,b,c,d\}$, $A_2 = \{a,e\}$, $A_3 = \{f\}$

does not form a partition of S because $A_1 \cap A_2 \neq \emptyset$

b) $A_1 = \{a,b\}$, $A_2 = \{c,d\}$, $A_3 = \{e,f\}$

forms a partition of S, because sets are disjoint and their union is S.

c) $A_1 = \{a\}$, $A_2 = \{b,c,d\}$, $A_3 = \{e\}$

Section 9.5 *Equivalence relations*

Partitions

Example: Suppose that S = {a,b,c,d,e,f}. Which collections of sets form a partition of S?

a) A_1 = {a,b,c,d}, A_2 = {a,e}, A_3 = {f}

does not form a partition of S because $A_1 \cap A_2 \neq \emptyset$

b) A_1 = {a,b}, A_2 = {c,d}, A_3 = {e,f}

forms a partition of S, because sets are disjoint and their union is S.

c) A_1 = {a}, A_2 = {b,c,d}, A_3 = {e}

does not form a partition of S because $A_1 \cup A_2 \cup A_3 \neq S$

Section 9.5 *Equivalence relations*

Partitions and equivalence relations

[Theorem] Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has sets A_i, $i \in I$, as its equivalence classes.

No proof. It is a summary of all the connections we have established between equivalence relations and partitions.

Section 9.5 *Equivalence relations*

Partitions and equivalence relations

Example: List the ordered pairs in the equivalence relation R produced by partition $A_1 = \{1, 2\}$, $A_2 = \{3\}$, and $A_3 = \{4, 5\}$ of the set $S = \{1, 2, 3, 4, 5\}$.

Section 9.5 *Equivalence relations*

Partitions and equivalence relations

Example: List the ordered pairs in the equivalence relation R produced by partition $A_1 = \{1, 2\}$, $A_2 = \{3\}$, and $A_3 = \{4, 5\}$ of the set $S = \{1, 2, 3, 4, 5\}$.

Solution:

$A_1 = \{1, 2\}$ is an equivalence class of R, therefore pair $(a,b) \in R$ iff $a,b \in A_1$.

Section 9.5 *Equivalence relations*

Partitions and equivalence relations

Example: List the ordered pairs in the equivalence relation R produced by partition A_1 = {1, 2}, A_2 = {3}, and A_3 = {4, 5} of the set S = {1, 2, 3, 4, 5}.

Solution:

A_1 = {1, 2} is an equivalence class of R, therefore pair (a,b) $\in R$ iff a,b $\in A_1$. So, (1,2), (2,1), (1,1), and (2,2) $\in R$.

Partitions and equivalence relations

Example: List the ordered pairs in the equivalence relation R produced by partition $A_1 = \{1, 2\}$, $A_2 = \{3\}$, and $A_3 = \{4, 5\}$ of the set $S = \{1, 2, 3, 4, 5\}$.

Solution:

$A_1 = \{1, 2\}$ is an equivalence class of R, therefore pair $(a,b) \in R$ iff $a,b \in A_1$. So, $(1,2)$, $(2,1)$, $(1,1)$, and $(2,2) \in R$.

$A_2 = \{3\}$ is an equivalence class of R, therefore pair $(a,b) \in R$ iff $a,b \in A_2$.

Section 9.5 *Equivalence relations*

Partitions and equivalence relations

Example: List the ordered pairs in the equivalence relation R produced by partition $A_1 = \{1, 2\}$, $A_2 = \{3\}$, and $A_3 = \{4, 5\}$ of the set $S = \{1, 2, 3, 4, 5\}$.

Solution:

$A_1 = \{1, 2\}$ is an equivalence class of R, therefore pair $(a,b) \in R$ iff $a,b \in A_1$. So, $(1,2)$, $(2,1)$, $(1,1)$, and $(2,2) \in R$.

$A_2 = \{3\}$ is an equivalence class of R, therefore pair $(a,b) \in R$ iff $a,b \in A_2$. So, $(3,3) \in R$.

$A_3 = \{4, 5\}$ is an equivalence class of R, therefore pair $(a,b) \in R$ iff $a,b \in A_3$.

Section 9.5 *Equivalence relations*

Partitions and equivalence relations

Example: List the ordered pairs in the equivalence relation R produced by partition $A_1 = \{1, 2\}$, $A_2 = \{3\}$, and $A_3 = \{4, 5\}$ of the set $S = \{1, 2, 3, 4, 5\}$.

Solution:

$A_1 = \{1, 2\}$ is an equivalence class of R, therefore pair $(a,b) \in R$ iff $a,b \in A_1$. So, $(1,2)$, $(2,1)$, $(1,1)$, and $(2,2) \in R$.

$A_2 = \{3\}$ is an equivalence class of R, therefore pair $(a,b) \in R$ iff $a,b \in A_2$. So, $(3,3) \in R$.

$A_3 = \{4, 5\}$ is an equivalence class of R, therefore pair $(a,b) \in R$ iff $a,b \in A_3$. So, $(4,5)$, $(5,4)$, $(4,4)$, and $(5,5) \in R$.

Section 9.5 *Equivalence relations*

Partitions and equivalence relations

Example: List the ordered pairs in the equivalence relation R produced by partition $A_1 = \{1, 2\}$, $A_2 = \{3\}$, and $A_3 = \{4, 5\}$ of the set $S = \{1, 2, 3, 4, 5\}$.

Solution:

$A_1 = \{1, 2\}$ is an equivalence class of R, therefore pair $(a,b) \in R$ iff $a,b \in A_1$. So, $(1,2)$, $(2,1)$, $(1,1)$, and $(2,2) \in R$.

$A_2 = \{3\}$ is an equivalence class of R, therefore pair $(a,b) \in R$ iff $a,b \in A_2$. So, $(3,3) \in R$.

$A_3 = \{4, 5\}$ is an equivalence class of R, therefore pair $(a,b) \in R$ iff $a,b \in A_3$. So, $(4,5)$, $(5,4)$, $(4,4)$, and $(5,5) \in R$.

$R = \{(1,1),(2,2),(3,3),(4,4),(5,5),(1,2),(2,1),(4,5),(5,4)\}$.