Section 9.2 *n-ary relations*

Why do we need to study relations?

Relations are used to solve problems involving
communications networks,
project scheduling, and
identifying elements in sets with common properties.

Section 9.2 *n-ary relations*

The theory of relational databases rest on *n-ary relations*

[Def] Let $A_1, A_2, \ldots, A_n$ be sets. An *n-ary relation* on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$.
The sets are called the *domains* of the relation, and
n is called its *degree*.

Section 9.2 *n-ary relations*

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Example: Let *R* be a relation on $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times Z$ consisting of 4-tuples that form arithmetic progression (sequence)
That is $(a,b,c,d) \in R$ if and only if (iff) $\exists k \in Z$ such that $b = a+k$, $c = a+2k$, $d = 2+3k$.
Which of the following tuples belong to the relation *R*?
(1,0,2,3) (2,1,0,-1) (3,7,9,11) (1,3,9,27)

Section 9.2 *n-ary relations*

The theory of relational databases rest on *n-ary relations*

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(1,0,2,3) (2,1,0,-1) (3,7,9,11) (1,3,9,27)

Section 9.2 *n-ary relations*

Databases and relations

Time is money!

The time required to manipulate information in a database depends on how this information is stored.

The operations of:
- adding/deleting/updating records,
- searching for records,
- combining records from overlapping databases

are performed million of times each day is a large database.

Various methods for representing databases have been developed. We will discuss one of them: **relational data model**.

Section 9.2 *n-ary relations*

Relational Model

Relational data model:

Activity Code	Activity Name
23	Patching
24	Overlay
25	Crack Sealing

- all data is represented in terms of tuples, grouped into relations

Key = 24

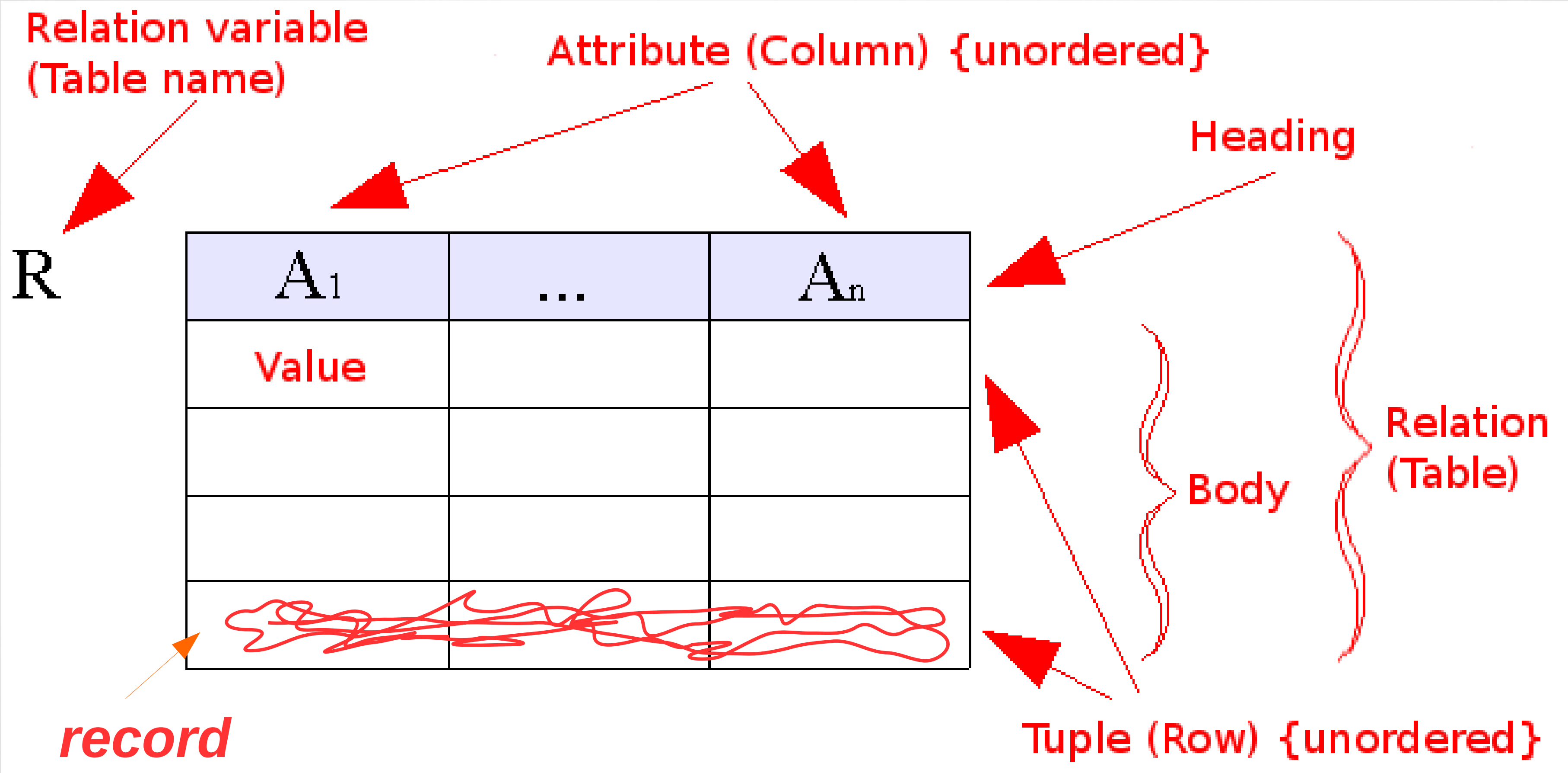
Activity Code	Date	Route No.
24	01/12/01	I-95
24	02/08/01	I-66

Date	Activity Code	Route No.
01/12/01	24	I-95
01/15/01	23	I-495
02/08/01	24	I-66

- a database organized in terms of the relational model is a relational database

Section 9.2 *n-ary relations*

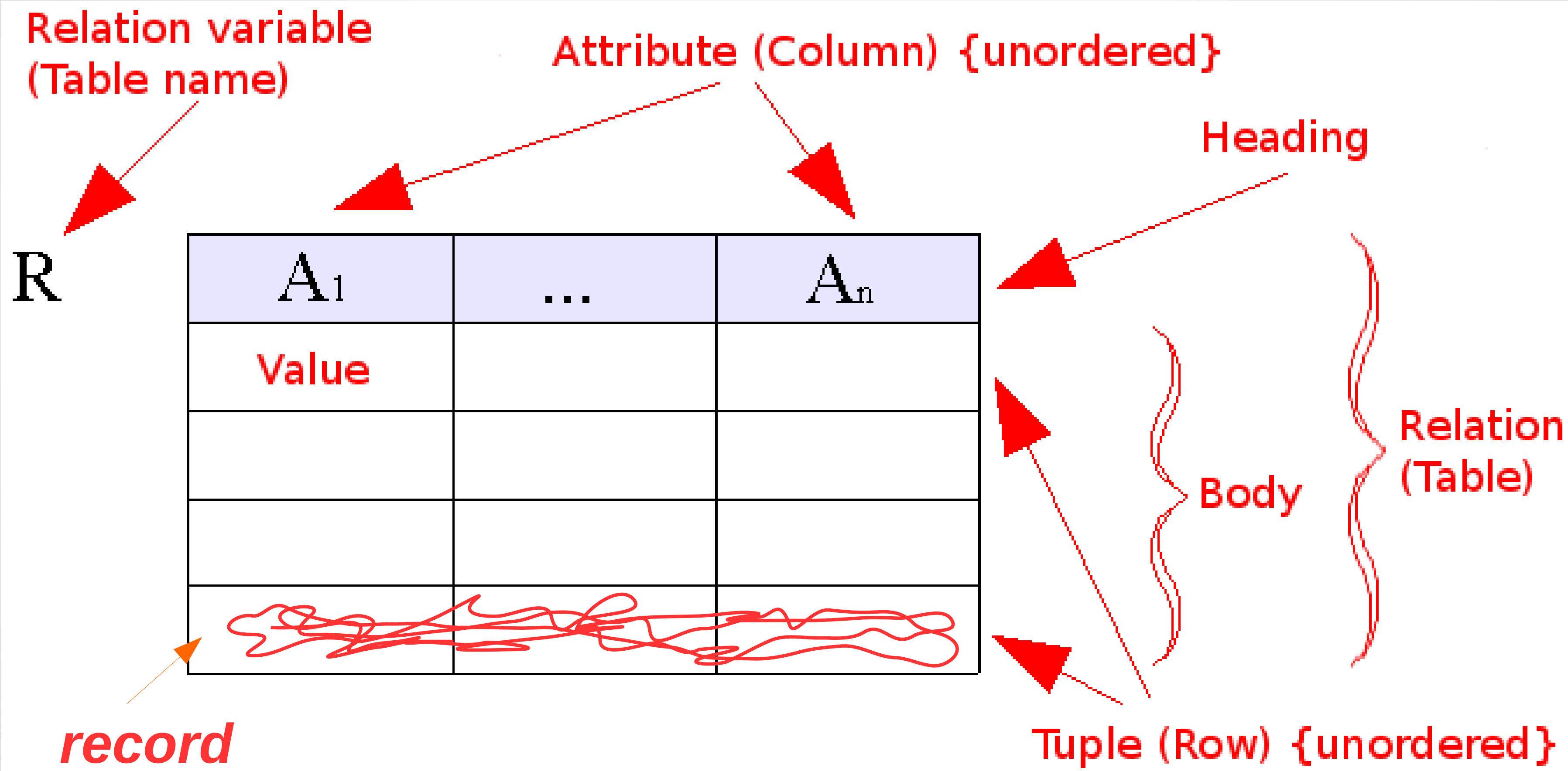
Relational data model:



Database consists of ***records***, which are *n-tuples*, made up of ***fields***.

Section 9.2 *n-ary relations*

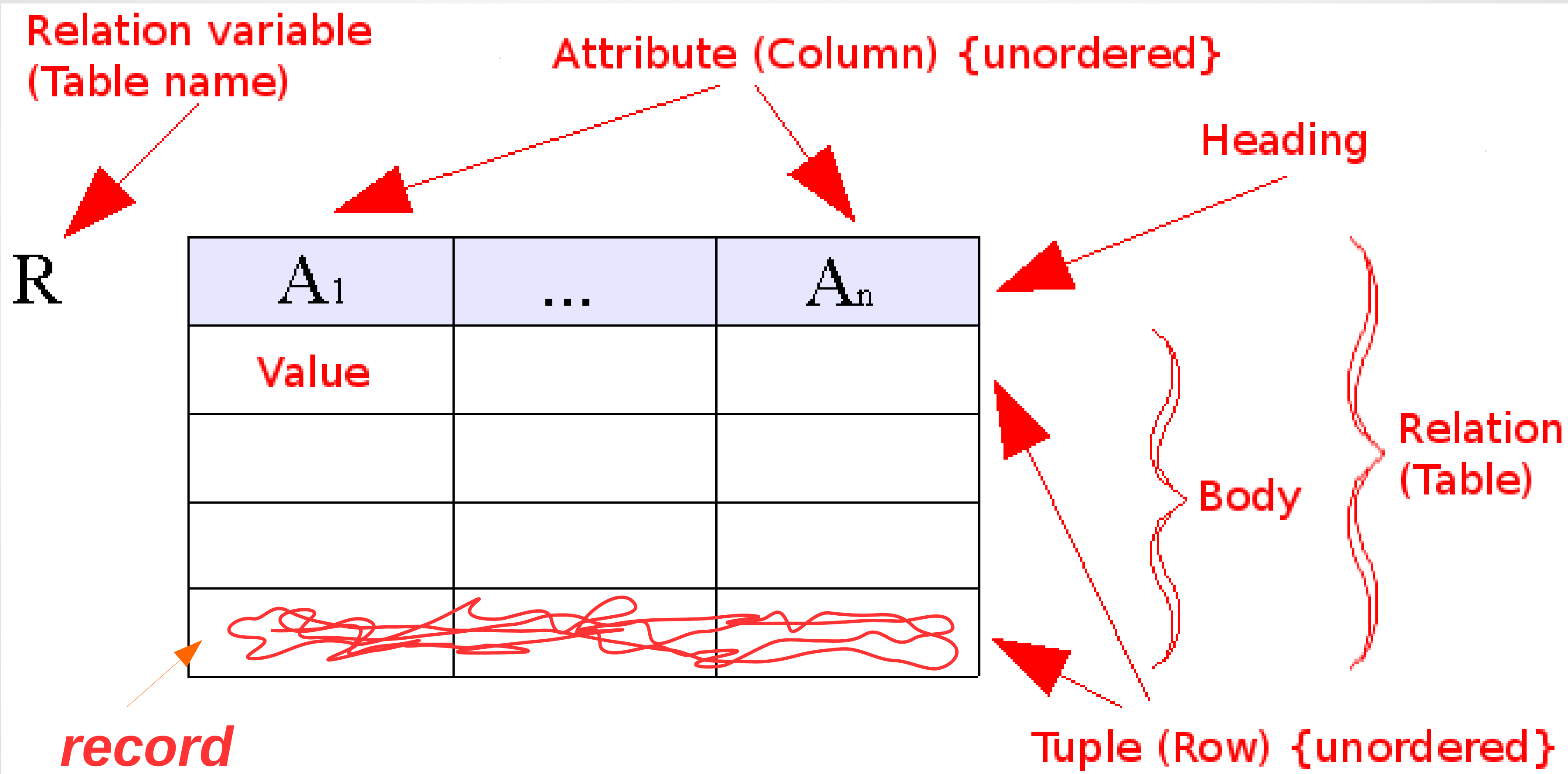
Relational data model:



Relations used to represent databases are also called ***tables***, because they are often displayed as tables.

Section 9.2 *n-ary relations*

Relational data model:



Each column corresponds to an ***attribute*** of the database.

Section 9.2 *n-ary relations*

Relational data model:

Level1 Level2 Level3 Level4 Transaction Table

Region	Country	Product	ID
Europe	Russia	Barley	1
Asia	India	Dry Bean	2
Europe	Turkey	Apricot	3
Northern America	USA	Almond	4
Asia	Indonesia	Cinnamon	5
Central america	Mexico	Avacado	6
Asia	India	Millet	7
Europe	Turkey	Fig	8
Asia	India	Lemon	9
Northern America	USA	Soybean	10

ID	Price
1	23
2	13
3	34
4	53
5	34
6	43
7	36
8	77
9	80
10	83

A domain is a ***primary key*** when no two tuples have the same value from this domain.

Section 9.2 *n-ary relations*

Relational data model:

Records are often added to or deleted from databases.

A <u>primary key</u> should be chosen that <u>remains one</u> whenever the database is changed.

The current collection of n-tuples in a relation is called the ***extension*** of the relation.

The more permanent part of the database, including the name and attributes of the database, is called its ***intension***.

<u>When selecting a primary key</u>: choose the one that can serve as a primary key for all possible extensions of the database.

Section 9.2 *n-ary relations*

Relational data model:

intension

extension

Students			
Name	**ID**	**Major**	**GPA**
Alex Smith	19284	CS	3.96
Jack Frost	14665	ART	3.99
Alex Smith	10984	MATH	3.89

ID is a primary key

Section 9.2 *n-ary relations*

Relational data model:

Students			
Name	**ID**	**Major**	**Course**
Alex Smith	19284	CS	CSI 32
Jack Frost	14665	ART	ENG 12
Alex Smith	19284	CS	CSI 35
Alex Smith	19284	CS	CHM 12
John Gold	87543	MATH	MTH 32
John Gold	87543	MATH	CSI 35

ID can no longer serve as a primary key

Section 9.2 *n-ary relations*

Relational data model:

Students			
Name	**ID**	**Major**	**Course**
Alex Smith	19284	CS	CSI 32
Jack Frost	14665	ART	ENG 12
Alex Smith	19284	CS	CSI 35
Alex Smith	19284	CS	CHM 12
John Gold	87543	MATH	MTH 32
John Gold	87543	MATH	CSI 35

Combinations of domains can also uniquely identify n-tuples in an n-ary relation: ***composite key***

Section 9.2 *n-ary relations*

Relational data model:

Students			
Name	**ID**	**Major**	**Course**
Alex Smith	19284	CS	CSI 32
Jack Frost	14665	ART	ENG 12
Alex Smith	19284	CS	CSI 35
Alex Smith	19284	CS	CHM 12
John Gold	87543	MATH	MTH 32
John Gold	87543	MATH	CSI 35

ID x **Course** make up ***composite key***

Section 9.2 *n-ary relations*

Operations on *n-ary* relations:

Section 9.2 *n-ary relations*

Operations on *n-ary* relations:

Let R be a relation of degree n

selection operator s_C, where C is a condition that elements (tuples) of R must satisfy, maps the relation R to the relation of the same degree each tuples of which satisfy the condition C.

Students

Name	**ID**	**Major**	**GPA**
Alex Smith	19284	CS	3.96
Jack Frost	14665	ART	3.99
Alex Smith	10984	MATH	3.89
Maria DeSoto	26846	NUR	3.99
Clarissa Marc	37584	CS	3.97

Section 9.2 *n-ary relations*

Operations on *n-ary* relations:

Example:
Students is a relation of degree *4*

Let C_1 be "Major = CS"
Then s_{C1}, will produce the relation

Students			
Name	**ID**	**Major**	**GPA**
Alex Smith	19284	CS	3.96
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Section 9.2 *n-ary relations*

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Students			
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Section 9.2 *n-ary relations*

Operations on *n-ary* relations:

Let R be a relation of degree n

projection operator P produces a new relation of degree m, $m \leq n$ by deleting the same fields in every record of the relation R.

Students			
Name	**ID**	**Major**	**GPA**
Alex Smith	19284	CS	3.96
Jack Frost	14665	ART	3.99
Alex Smith	10984	MATH	3.89
Maria DeSoto	26846	NUR	3.99
Clarissa Marc	37584	CS	3.97

Section 9.2 *n-ary relations*

Operations on *n-ary* relations:

Example:
Students is a relation of degree *4*

Find $P_{1,4}$ and $P_{1,2,3}$

Students			
Name	**ID**	**Major**	**GPA**
Alex Smith	19284	CS	3.96
Jack Frost	14665	ART	3.99
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Clarissa Marc	37584	CS	3.97

Section 9.2 *n-ary relations*

Operations on *n-ary* relations:

Example:
Students is a relation of degree *4*

$P_{1,4}$

Students	
Name	**GPA**
Alex Smith	3.96
Jack Frost	3.99
Alex Smith	3.89
Maria DeSoto	3.99
Clarissa Marc	3.97

Section 9.2 *n-ary relations*

Operations on *n-ary* relations:

Example:
Students is a relation of degree *4*

$P_{1,2,3}$

Students		
Name	**ID**	**Major**
Alex Smith	19284	CS
Jack Frost	14665	ART
Alex Smith	10984	MATH
Maria DeSoto	26846	NUR
Clarissa Marc	37584	CS

Section 9.2 *n-ary relations*

Operations on *n-ary* relations:

Let *R* be a relation of degree *n* , let *S* be a relation of degree *m*.

The ***join operation*** allows to combine two relations (tables) into one when these tables share some identical fields.

$J_p(R,S)$

Section 9.2 *n-ary relations*

Operations on *n-ary* relations:

R				
Name	**ID**	**status**	**Major**	**GPA**
Alex Smith	19284	FR	CS	3.96
Jack Frost	14665	SO	ART	3.99
Alex Smith	10984	SO	MATH	3.89
Maria DeSoto	26846	FR	NUR	3.99
Clarissa Marc	37584	SO	CS	3.97

Find $J_{1,2}(Status, Students)$

Section 9.2 *n-ary relations*

Operations on *n-ary* relations:

Status			Students			
Name	**ID**	**status**	**Name**	**ID**	**Major**	**GPA**
Alex Smith	19284	FR	Alex Smith	19284	CS	3.96
Jack Frost	14665	SO	Jack Frost	14665	ART	3.99
Alex Smith	10984	SO	Alex Smith	10984	MATH	3.89
Maria DeSoto	26846	FR	Maria DeSoto	26846	NUR	3.99
Clariss a Marc	37584	SO	Clarissa Marc	37584	CS	3.97

Find $J_{1,2}(Status,Students)$

Section 9.2 *n-ary relations*

Operations on *n-ary* relations:

Status		
Name	**ID**	**status**
Alex Smith	19284	FR
Jack Frost	14665	SO
Alex Smith	10984	SO
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Find $J_{1,2}(Status,Students)$

Matrices

*[**Def**] A matrix is a rectangular array of numbers. A matrix with m rows and n columns is called $m \times n$ matrix.*

plural form: matrices

denotaion: $\mathbf{A}_{m \times n}$

$$\mathbf{A}_{4 \times 3} = \begin{bmatrix} 1 & 2 & 6 \\ -2 & 0 & 11 \\ 9 & 1 & 7 \\ 2 & 19 & 3 \end{bmatrix}$$

matrix with $m = n$ is called square matrix

Two matrices are equal is they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

Section 9.3 *Representing relations*

Matrices

a_{ij} – element of matrix in row *i* and column *j*

	1st	2nd	3rd	4th	
	-9	2	7	10	1st
A =	1	3	5	2	2nd
	2	-8	3	11	3rd

$$\mathbf{A} = \begin{bmatrix} -9 & 2 & 7 & 10 \\ 1 & 3 & 5 & 2 \\ 2 & -8 & 3 & 11 \end{bmatrix}$$

A has 3 rows, 4 columns

$a_{23} = 5$ — 2nd row, 3rd column

$a_{14} = 10$ — 1st row, 4th column

Section 9.3 *Representing relations*

Matrices

a_{ij} – element of matrix in row *i* and column *j*

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{22} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

(column label: *j*; row label: *i*)

In computer programs matrices can be used to represent the relations.

Section 9.3 *Representing relations*

Representing relations using matrices

Zero-one matrices can be used to represent relations.

Assume relation $R : A \to B$, where $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, then R can be represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R, \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$

Section 9.3 *Representing relations*

Representing relations using matrices

Example: Let $A = \{1,2,3,4\}$ and $B = \{1,2,3\}$, and let $R = \{ (a,b) \mid a \in A,\ b \in B, \text{ and } a < b\}$. Give the matrix representing relation R.

Section 9.3 *Representing relations*

Representing relations using matrices

Example: Let A = {1,2,3,4} and B = {1,2,3}, and let R = { (a,b) | a $\in$ A, b $\in$ B, and a < b} . Give the matrix representing relation R.

Solution: R = {(1,2), (1,3), (2,3)}

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} R \quad 1 \quad 2 \quad 3 \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Section 9.3 *Representing relations*

Representing relations using matrices

Example 2: Let A = {1,2,3} and B = {a,b,c,d,e}. Which ordered pairs are in the relation *R* represented by the matrix?

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Section 9.3 *Representing relations*

Representing relations using matrices

Example 2: Let A = {1,2,3} and B = {a,b,c,d,e}. Which ordered pairs are in the relation *R* represented by the matrix?

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Answer: R = {(1,b), (1,c), (1,d), (2,a), (2,d), (3,b), (3,e)}

Section 9.3 *Representing relations*

Representing relations using matrices

Matrix of a relation on a set is a square matrix.

the *relation is reflexive* iff its matrix has all 1's on diagonal

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Section 9.3 *Representing relations*

Representing relations using matrices

Matrix of a relation on a set is a square matrix.

the *relation is symmetric* iff its matrix is symmetric.
The *matrix is symmetric* when it is equal to its transpose.

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} = M_R^T = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Transpose of a matrix

[**Def**] Let $\mathbf{A}_{m\times n} = [a_{i\,j}]$. The *transpose* of **A** is the $n \times m$ matrix obtained by interchanging the rows and the columns of **A**.

denotation: $\mathbf{A}^t$

Example:

Let $\mathbf{A} = \begin{bmatrix} 1 & 4 & 6 & 8 \\ 2 & -2 & 5 & -9 \\ 2 & 0 & 5 & -1 \end{bmatrix}$

Find $\mathbf{A}^t$.

Section 9.3 *Representing relations*

Transpose of a matrix

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Example:

Let $\mathbf{A} = \begin{bmatrix} 1 & 4 & 6 & 8 \\ 2 & -2 & 5 & -9 \\ 2 & 0 & 5 & -1 \end{bmatrix}$

Find $\mathbf{A}^t$.

<u>Solution</u>:

$\mathbf{A}^t = \begin{bmatrix} 1 & & \\ 4 & & \\ 6 & & \\ 8 & & \end{bmatrix}$

Section 9.3 *Representing relations*

Transpose of a matrix

[**Def**] Let $\mathbf{A}_{m\times n} = [a_{i\,j}]$. The *transpose* of **A** is the $n \times m$ matrix obtained by interchanging the rows and the columns of **A**.

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Example:

Let $\mathbf{A} = \begin{bmatrix} 1 & 4 & 6 & 8 \\ 2 & -2 & 5 & -9 \\ 2 & 0 & 5 & -1 \end{bmatrix}$

Find $\mathbf{A}^t$.

Solution:

$\mathbf{A}^t = \begin{bmatrix} 1 & 2 \\ 4 & -2 \\ 6 & 5 \\ 8 & -9 \end{bmatrix}$

Section 9.3 *Representing relations*

Transpose of a matrix

[**Def**] Let $\mathbf{A}_{m\times n} = [a_{i\,j}]$. The *transpose* of **A** is the $n \times m$ matrix obtained by interchanging the rows and the columns of **A**.

denotation: $\mathbf{A}^t$

Example:

Let $\mathbf{A} = \begin{bmatrix} 1 & 4 & 6 & 8 \\ 2 & -2 & 5 & -9 \\ 2 & 0 & 5 & -1 \end{bmatrix}$

Find $\mathbf{A}^t$.

Solution:

$$\mathbf{A}^t = \begin{bmatrix} 1 & 2 & 2 \\ 4 & -2 & 0 \\ 6 & 5 & 5 \\ 8 & -9 & -1 \end{bmatrix}$$

Section 9.3 *Representing relations*

Representing relations using matrices

Example: For relations *R* and *S* on set A={1,2,3,4} that are represented by their corresponding matrices. Check whether the relations are *reflexive* and/or *symmetric*.

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Section 9.3 *Representing relations*

Representing relations using matrices

Example: For relations *R* and *S* on set A={1,2,3,4} that are represented by their corresponding matrices. Check whether the relations are *reflexive* and/or *symmetric*.

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

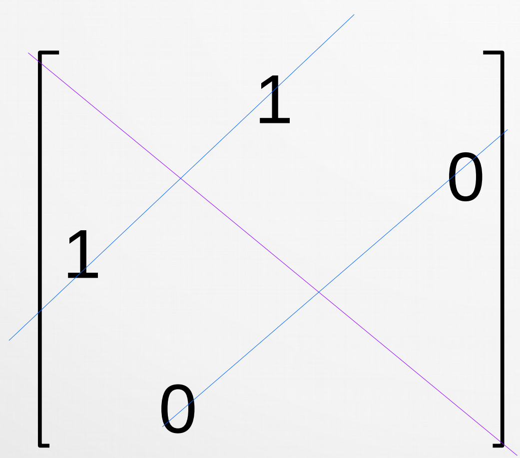
symmetric *reflexive*

Section 9.3 *Representing relations*

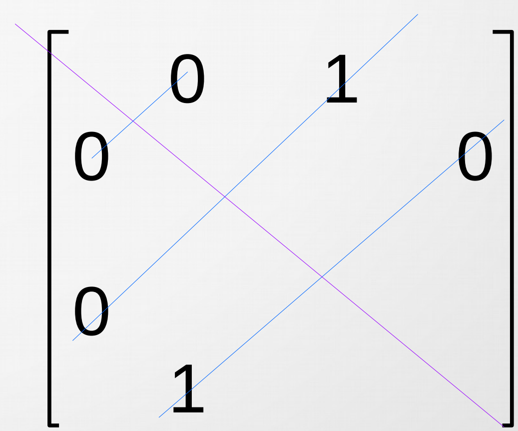
Representing relations using matrices

Matrix of a relation on a set is a square matrix.

the *relation is symmetric* iff its matrix is symmetric.
the *relation is antisymmetric* if either $m_{ij} = 0$ or $m_{ji} = 0$ for $i \neq j$.



symmetric



antisymmetric

Section 9.3 *Representing relations*

Representing relations using matrices

Example: For relations *R* and *S* on set A={1,2,3,4} that are represented by their corresponding matrices. Check whether the relations are *symmetric* and/or *antisymmetric*.

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Section 9.3 *Representing relations*

Representing relations using matrices

Example: For relations *R* and *S* on set A={1,2,3,4} that are represented by their corresponding matrices. Check whether the relations are *symmetric* and/or *antisymmetric*.

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

symmetric *antisymmetric*

Section 9.3 *Representing relations*

Representing relations using matrices

We can also find *union* $\cup$ and *intersection* $\cap$ of two relations on the set.

Example: Let R_1 and R_2 be two relations on the set A, and they are represented by the matrices:

$$M_{R1} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad M_{R2} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Then $M_{R1 \cup R2} = M_{R1} \vee M_{R2}$ and $M_{R1 \cap R2} = M_{R1} \wedge M_{R2}$

Section 9.3 *Representing relations*

Representing relations using matrices

Example: Let R_1 and R_2 be two relations on the set A, and they are represented by the matrices:

$$M_{R1} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad M_{R2} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$M_{R1 \cup R2} = M_{R1} \vee M_{R2} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Section 9.3 *Representing relations*

Representing relations using matrices

Example: Let R_1 and R_2 be two relations on the set A, and they are represented by the matrices:

$$M_{R1} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad M_{R2} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$M_{R1 \cap R2} = M_{R1} \wedge M_{R2} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Section 9.3 *Representing relations*

Representing relations using matrices

We can find composition $R \circ S$ of relations on the set as well.

Example: Let R and S be two relations on the set A, and they are represented by the matrices:

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Then $M_{R \circ S} = M_R \odot M_S$ Boolean product of matrices

Section 9.3 *Representing relations*

product of matrices

[Def] Let **A** be $m \times k$ matrix and **B** $k \times n$ matrix.

The product of **A** and **B**, **A·B** is the $m \times n$ matrix with its $(i,j)^{th}$ element/entry equal to the sum of the products of the corresponding elements from the i^{th} row of **A** by the j^{th} column of **B**, i.e.

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Example: Multiply matrices

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix} =$$

3×2 $\quad$ 2×2

Section 9.3 *Representing relations*

product of matrices

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$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 6 & 9 \\ -3 & -4 \end{bmatrix}$$

3×2 $\quad$ 2×2 $\quad$ 3×2

Boolean product of matrices

Similar to the product of matrices, but all operations are **modulo** 2

Example: Find Boolean product:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

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