Recursive Definitions and Structural induction practice

Example 1: Find $f(2)$, $f(3)$, $f(4)$ and $f(5)$ is f is defined recursively by:

$$f(0) = f(1) = 1, \text{ and}$$

$$f(n+1) = f(n) / f(n-1), \quad \text{for } n = 1,2,\ldots$$

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$$f(n+1) = f(n) / f(n-1), \quad \text{for } n = 1,2,\ldots$$

Solution:

$f(2) = f(1) / f(0) = 1 / 1 = 1$
$f(3) = f(2) / f(1) = 1 / 1 = 1$
$f(4) = f(3) / f(2) = 1 / 1 = 1$
$f(5) = f(4) / f(3) = 1 / 1 = 1$

Answer: $f(2) = f(3) = f(4) = f(5) = 1$

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Example 2: Determine whether each of these proposed definitions is a valid recursive definition of a function $f: W \rightarrow Z$. If f is **well defined**, find a non-recursive formula for $f(n)$ when $n \in W$.

a) $f(0) = 2, \quad f(1) = 3, \quad f(n) = f(n-2) - 2f(n-3)$ for $n \geq 2$

b) $f(0) = 0, \quad f(1) = 1, \quad f(n) = 2f(n-1)$ for $n \geq 2$

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a) $f(0) = 2, \quad f(1) = 3, \quad f(n) = f(n-2) - 2f(n-3)$ for $n \geq 2$

$f(2) = f(0) - 2f(-1)$, but f is not defined on -1.

Therefore, the definition is not valid

b) $f(0) = 0, \quad f(1) = 1, \quad f(n) = 2\,f(n-1)$ for $n \geq 2$

$f(0) = 0, \quad f(1) = 1, \quad f(2) = 2f(1) = 2, \quad f(3) = 2f(2) = 4,$

$f(4) = 2f(3) = 8, \quad f(5) = 2f(4) = 16, \quad f(6) = 2f(5) = 32$

Answer: $f(0) = 0$, and $f(n) = 2^{n-1}$, for $n \geq 1$

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Example 3: Give recursive definition of $S_m(n)$, the sum of the integer m and the nonnegative integer n.

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Example 3: Give recursive definition of $S_m(n)$, the sum of the integer *m* and the nonnegative integer *n*.

Answer:

$S_m(0) = m$

$S_m(n+1) = S_m(n) + 1$, for all integers n

Let's check:

$S_3(4) = 3+4 = 7$

$S_3(4) = S_3(3)+1 = (S_3(2)+1)+1 = S_3(2)+2 = (S_3(1)+1)+2 =$

$= S_3(1) + 3 = (S_3(0)+1)+3 = S_3(0) + 4 = 3 + 4 = 7$

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Example 4: Prove that $f_1^2+f_2^2+f_3^2 + \ldots + f_n^2 = f_n f_{n+1}$ for $n \in Z^+$, where f_n is the nth Fibonacci number.

Fibonacci numbers: $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, ...

Recursive Definitions and Structural induction practice

Example 4: Prove that $f_1^2+f_2^2+f_3^2 + \ldots + f_n^2 = f_n f_{n+1}$ for $n \in Z^+$, where f_n is the nth Fibonacci number.

Fibonacci numbers: $f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, \ldots$

Proof (by math. Induction):

Basis step: $n = 1$ $\quad f_1^2 = 1^2 = 1$ $\quad f_1 f_2 = 1 \cdot 1 = 1$

Therefore, $f_1^2 = f_1 f_2$

Inductive step: assume that for $k \in Z^+$,

$$f_1^2+f_2^2+f_3^2 + \ldots + f_k^2 = f_k f_{k+1} \qquad \text{(IH)}$$

Let's show that in this case $f_1^2+f_2^2+f_3^2 + \ldots + f_k^2 + f_{k+1}^2 = f_{k+1} f_{k+2}$:

$f_1^2+f_2^2+f_3^2 + \ldots + f_k^2 + f_{k+1}^2 = f_k f_{k+1} + f_{k+1}^2 = f_{k+1}(f_k + f_{k+1}) = f_{k+1} f_{k+2}$

This completes the inductive step.

By math. Induction we proved the formula above. q.e.d.

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Example 5: Give a recursive definition of the set of positive integer powers of 3.

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Example 5: Give a recursive definition of the set of positive integer powers of 3.

Answer:

Basis step: $3 \in S$

Recursive step: $3s \in S$, if $s \in S$

Let's check:

$3\ ,\ 3 \cdot 3 = 9,\ 9 \cdot 3 = 27,\ 27 \cdot 3 = 81, \ldots$

$3^1 \qquad 3^2, \qquad 3^3, \qquad 3^4$

Recursive Definitions and Structural induction practice

Example 6: Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0,0) \in S$

Recursive step: if $(a,b) \in S$,

then $(a+2,b+3) \in S$, and $(a+3,b+2) \in S$

a) List the elements of S produced by the first five applications of the recursive definition.

Recursive Definitions and Structural induction practice

Example 6: Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0,0) \in S$

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then $(a+2,b+3) \in S$, and $(a+3,b+2) \in S$

a) List the elements of S produced by the first five applications of the recursive definition.

(0,0)

(2,3), (3,2)

(4,6), (5,5), (6,4)

(6,9), (7,8), (8,7), (9,6)

(8,12), (9,11), (10,10), (11,9), (12,8)

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Example 6: Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0,0) \in S$

Recursive step: if $(a,b) \in S$,

then $(a+2,b+3) \in S$, and $(a+3,b+2) \in S$

b) Use strong induction on the number of applications of the recursive step of the def. to show that $5 \mid (a+b)$, if $(a,b) \in S$.

Let $P(n)$: "$5 \mid (a+b)$ if $(a+b)$ obtained by n applications of the rec. step"

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Example 6: Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0,0) \in S$
Recursive step: if $(a,b) \in S$,
then $(a+2,b+3) \in S$, and $(a+3,b+2) \in S$

b) Use strong induction on the number of applications of the recursive step of the def. to show that $5 \mid (a+b)$, if $(a,b) \in S$.

Let P(n): "$5 \mid (a+b)$ if $(a+b)$ obtained by n applications of the rec. step"
Basis step: P(0) is true, because (0,0) is obtained by 0 steps and $5|(0+0)$.
Inductive step: assume that $5|(s+t)$, whenever $(s,t) \in S$ is obtained by k or fewer applications of the rec. step. (IH)
Let's consider an element obtained by (k+1) applications of the rec. step: it was received from an element $(s,t) \in S$ obtained by k applications by either (s+2,t+3) or (s+3,t+2).
In the 1st case (s+2,t+3): s+2+t+3 = s+t+5. By IH $5|s+t$, and $5|5$, therefore $5|s+t+5$
In the 2nd case (s+3,t+2): s+3+t+2 = s+t+5, so similarly to the previous reasoning $5|s+t+5$
This completes inductive step.
By strong induction we proved that $5 \mid (a+b)$, if $(a,b) \in S$.
q.e.d.

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Example 6: Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0,0) \in S$

Recursive step: if $(a,b) \in S$,

then $(a+2,b+3) \in S$, and $(a+3,b+2) \in S$

c) Use structural induction to show that $5 \mid (a+b)$, if $(a,b) \in S$.

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Example 6: Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0,0) \in S$
Recursive step: if $(a,b) \in S$,
then $(a+2,b+3) \in S$, and $(a+3,b+2) \in S$

c) Use structural induction to show that $5 \mid (a+b)$, if $(a,b) \in S$.
Basis step: $(0,0) \in S$: $0+0 = 0$, 0 is divisible by 5, i.e. $5|0$
Recursive/Inductive step: assume we get a pair $(s,t) \in S$ after $k \geq 1$ applications of rec. step from def. and $5|s+t$. (IH)
Then for the next application of the rec. step there are two options :
Case 1: $(s+2,t+3) \in S$, in this case $(s+2)+(t+3) = s+t + 5$
$5|(s+t)$ (by IH), and $5|5$, therefore $5|(s+t+5)$
Case 2: $(s+3,t+2) \in S$, in this case $(s+3)+(t+2) = s+t + 5$
$5|(s+t)$ (by IH), and $5|5$, therefore $5|(s+t+5)$
This completes inductive step.
By structural induction we proved divisibility statement. q.e.d.