Plan

We will:

- Finish **Section 5.1** *Mathematical Induction*
- Cover **Section 5.2** *Strong induction and well-ordering*

Section 5.1 Mathematical Induction (continues)

Example: prove that $n^3 - n$ is divisible by 3 if $n \in \mathbf{Z}^+$

Section 5.1 Mathematical Induction (continues)

Example: prove that n^3-n is divisible by 3 if n $\in$ **Z**$^+$

Proof: Let P(n): “ n^3-n is divisible by 3 if n $\in$ **Z**$^+$ ”

Section 5.1 Mathematical Induction (continues)

Example: prove that n^3-n is divisible by 3 if $n \in \mathbf{Z}^+$

Proof: Let P(n): " n^3-n is divisible by 3 if $n \in \mathbf{Z}^+$ "

Base step: let n = 1, then

P(1): " 1^3-1 is divisible by 3 if $1 \in \mathbf{Z}^+$ "

Section 5.1 Mathematical Induction (continues)

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P(1): " 1^3-1 is divisible by 3 if $1 \in \mathbf{Z}^+$ " True!

0 is divisible by 3

Section 5.1 Mathematical Induction (continues)

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Inductive step: assume P(k) is true for any arbitrary fixed k, i.e. " k^3-k is divisible by 3 if $k \in \mathbf{Z}^+$ "

Let's show that in this case P(k+1) is also true:

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$(k+1)^3 – (k+1) = k^3+3k^2 + 3k +1 – k – 1 =$

Section 5.1 Mathematical Induction (continues)

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$=$

keep
(we will use it later)

Section 5.1 Mathematical Induction (continues)

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Let's show that in this case P(k+1) is also true:

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$= k^3 - k + 3(k^2 + k)$

divisible by IH

divisible by 3 because 3 is a factor

Section 5.1 Mathematical Induction (continues)

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hence their sum is divisible by 3 (see Theorem 1, page 238)

Section 5.1 Mathematical Induction (continues)

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divisible by 3 because 3 is a factor

thus we showed that P(k+1) also holds under the IH

Section 5.1 Mathematical Induction (continues)

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Section 5.1 Mathematical Induction (continues)

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Base step: let n = 1, then

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0 is divisible by 3

By mathematical induction we proved that n^3-n is divisible by 3 if $n \in \mathbf{Z}^+$

qed

Inductive step: assume P(k) is true for any arbitrary fixed k, i.e. " k^3-k is divisible by 3 if $k \in \mathbf{Z}^+$ " IH

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divisible by IH

divisible by 3 because 3 is a factor

thus we showed that P(k+1) also holds under the IH

This completes inductive step

Section 5.1 Mathematical Induction (continues)

Recommended:

See Example 10: proves that for any finite set S with n elements, the powerset P(S) has 2^n cardinality.

and

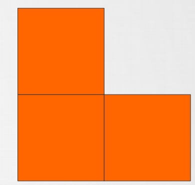
See Example 11 on page 323, which proves the generalization of one of De Morgan's laws about sets:

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

Section 5.1 Mathematical Induction (continues)

Example:

Let n be a positive integer. Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes, pieces shaped like the letter "L."



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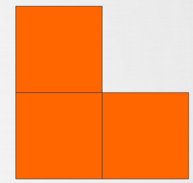

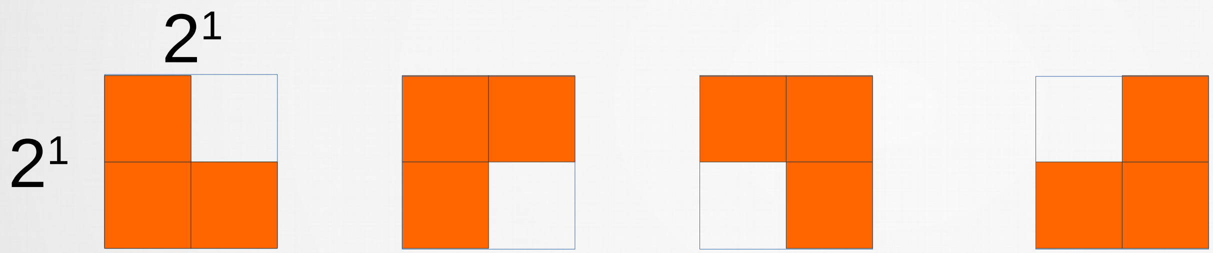

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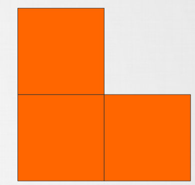
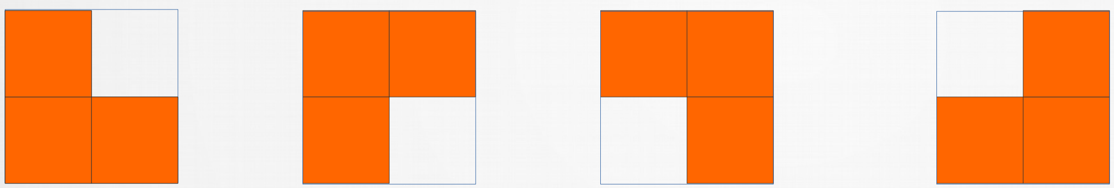


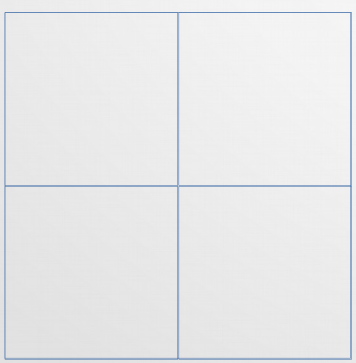
2^1

2^1



2^2

2^2



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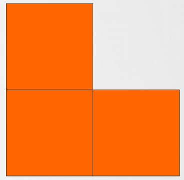

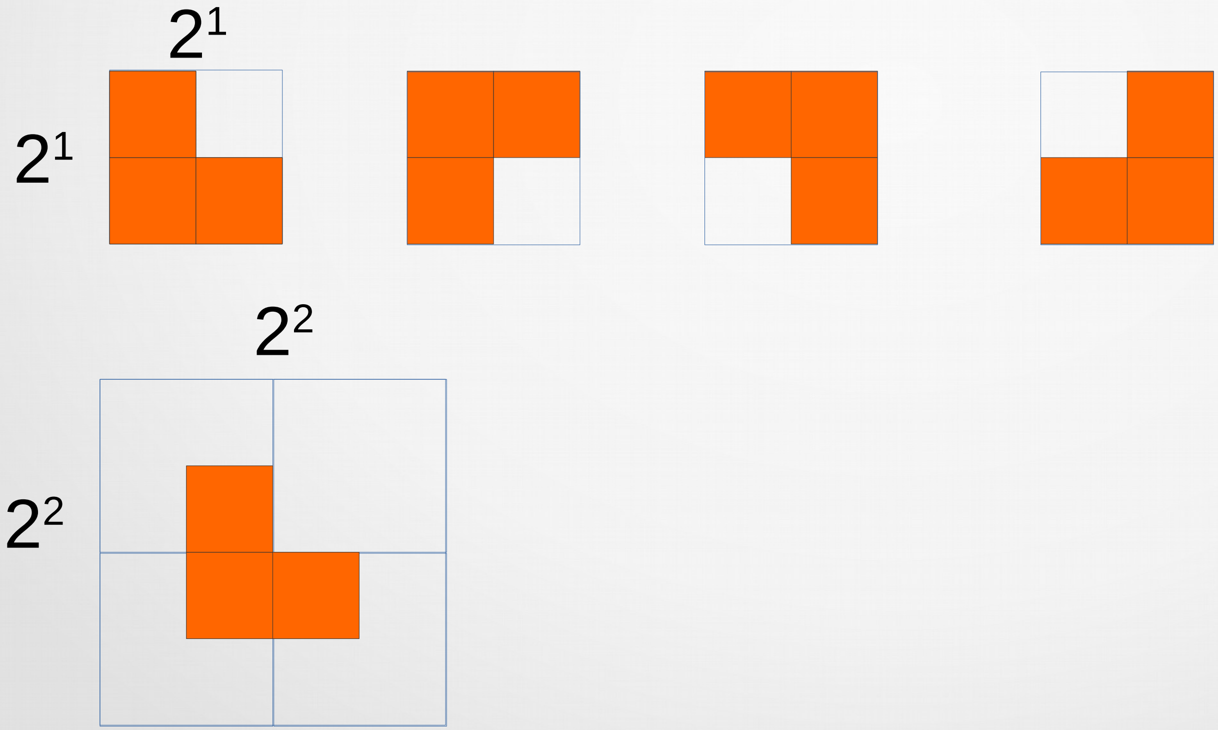

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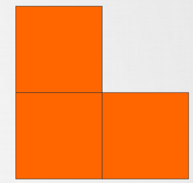

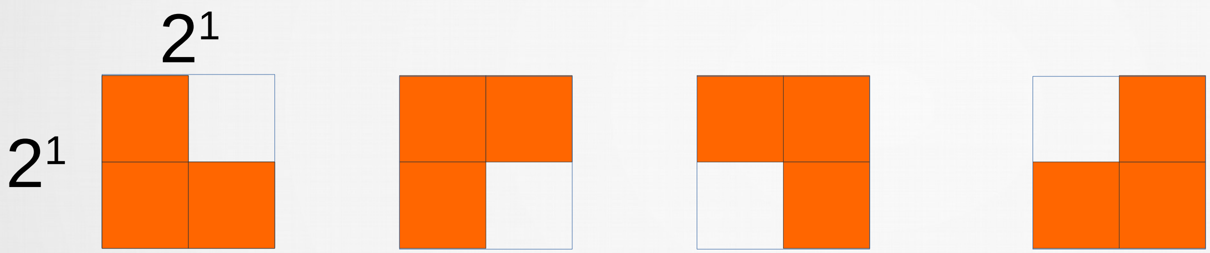

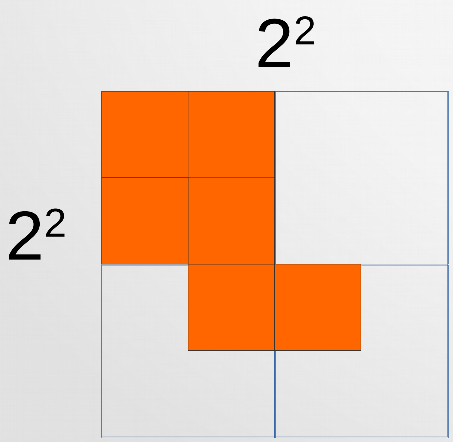

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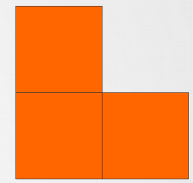

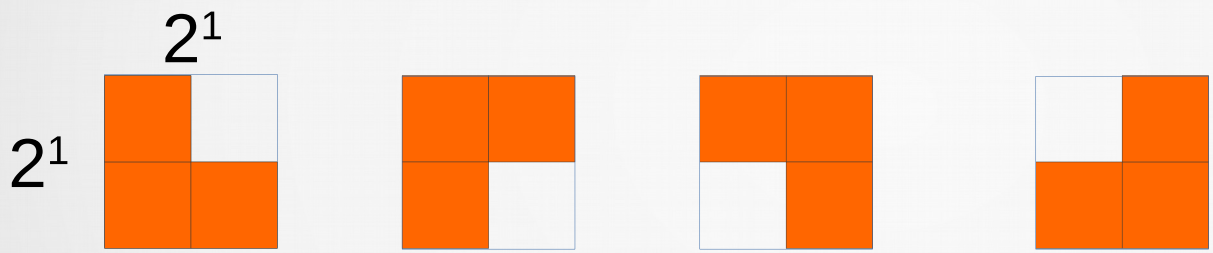

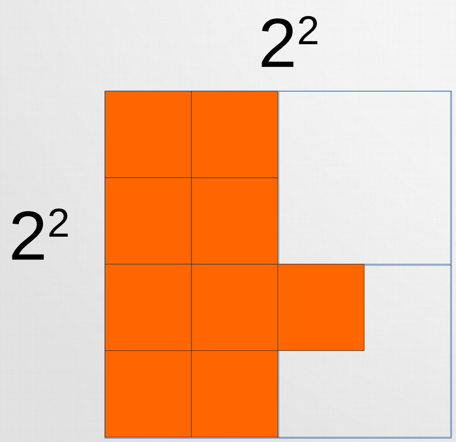

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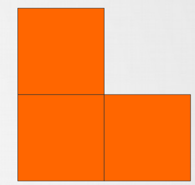

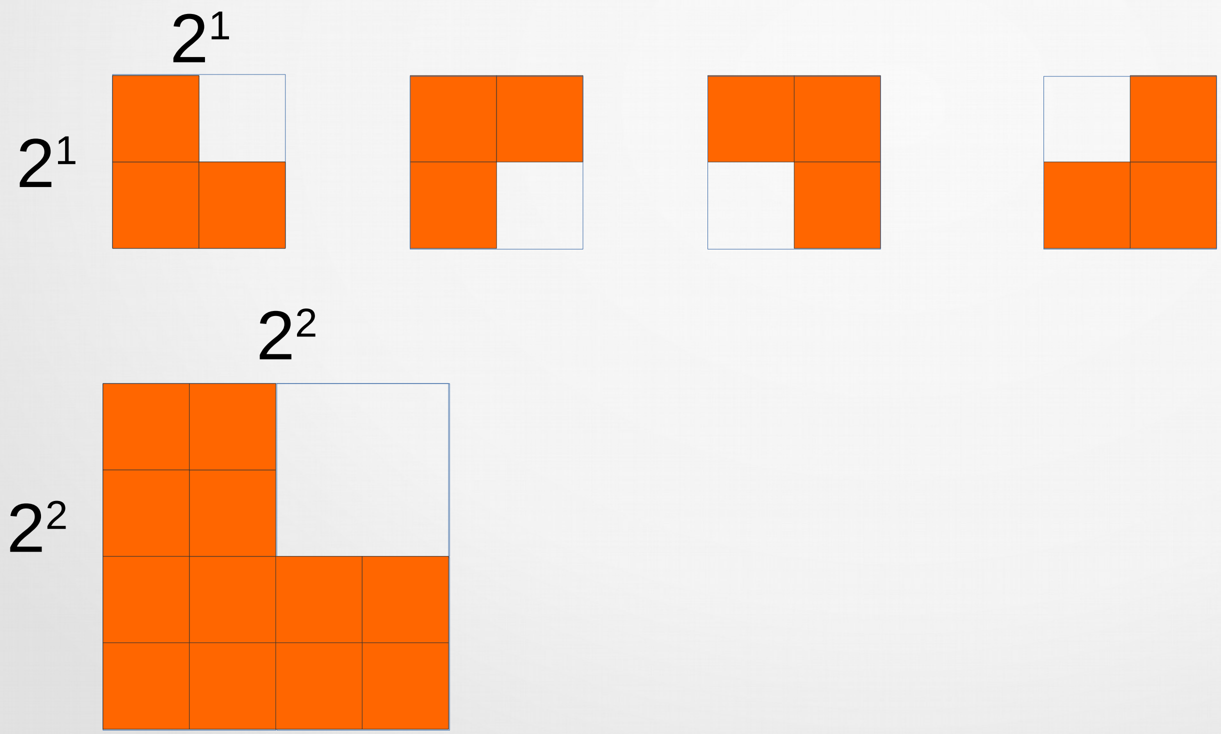

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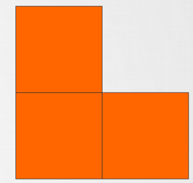

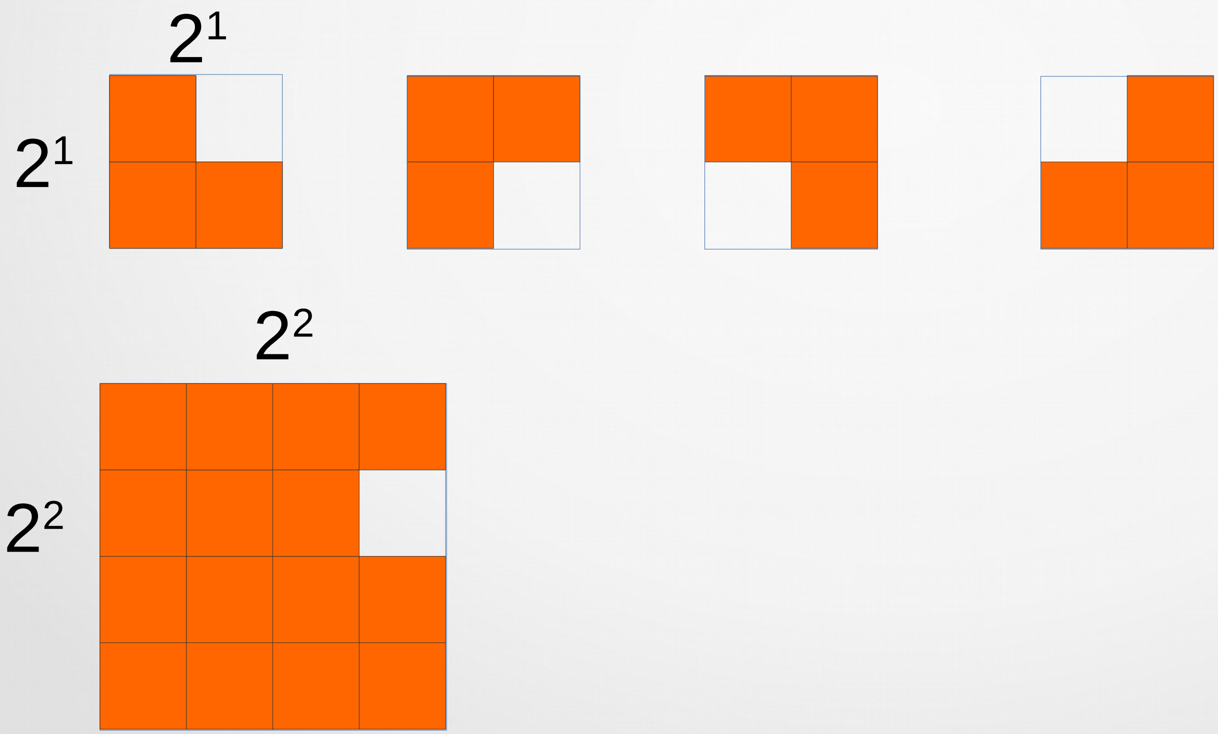

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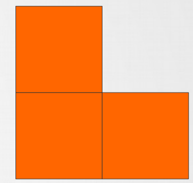
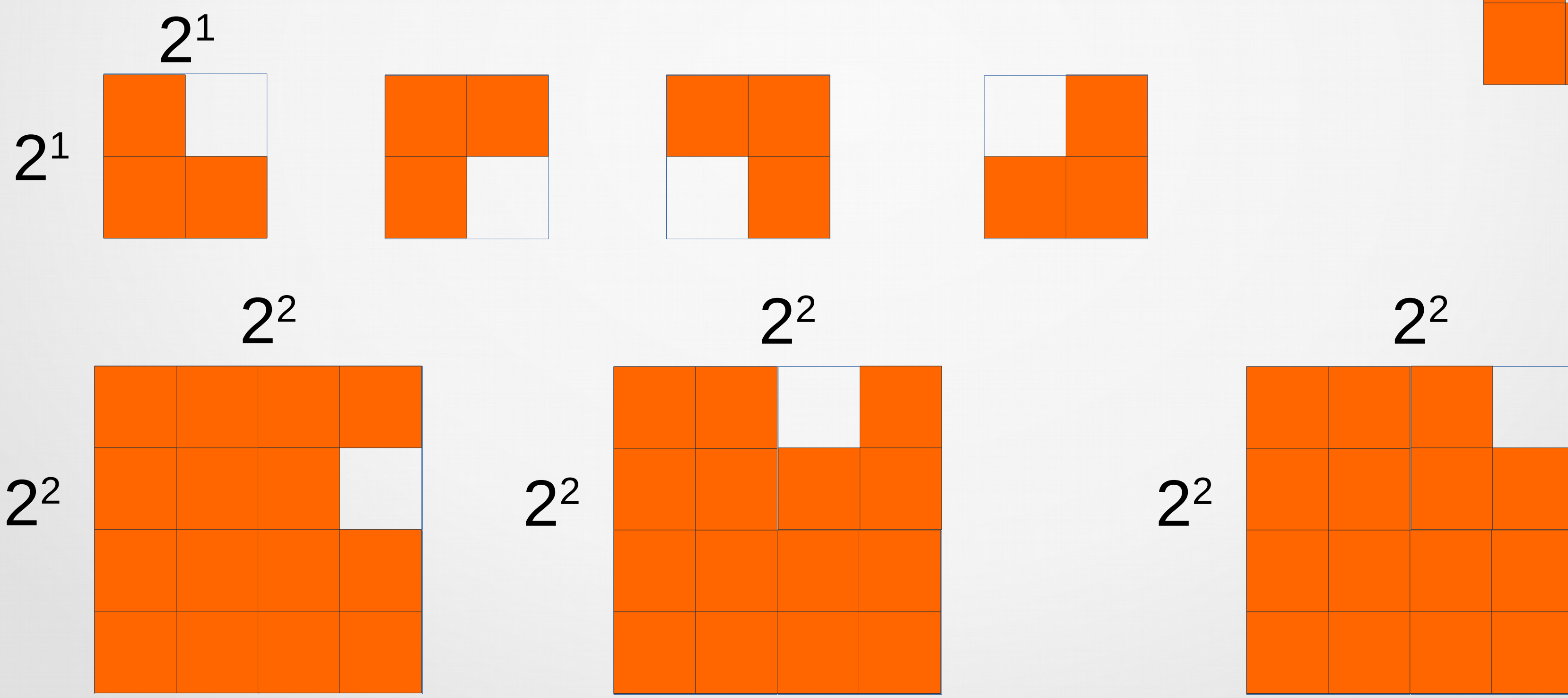




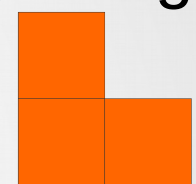
Note that any square can be removed from the checkerboard, we still can tile it!

Section 5.1 Mathematical Induction (continues)

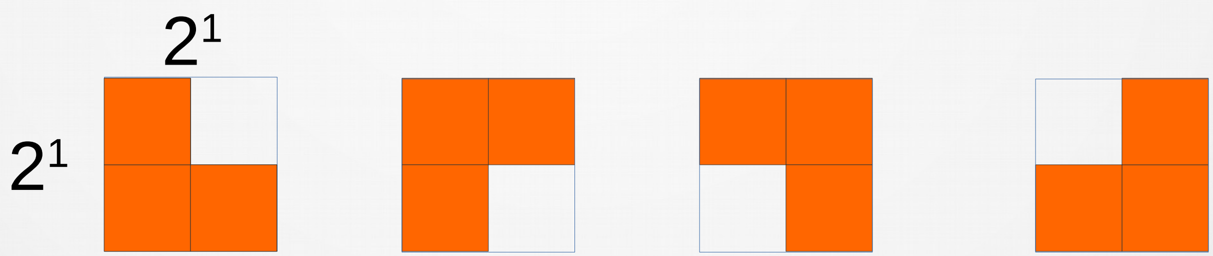
P(n)

Example:

Let n be a positive integer. Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes, pieces shaped like the letter "L."



Basis step: n = 1, hence we have 2×2 checkerboard



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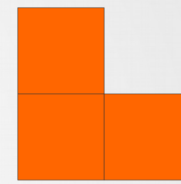
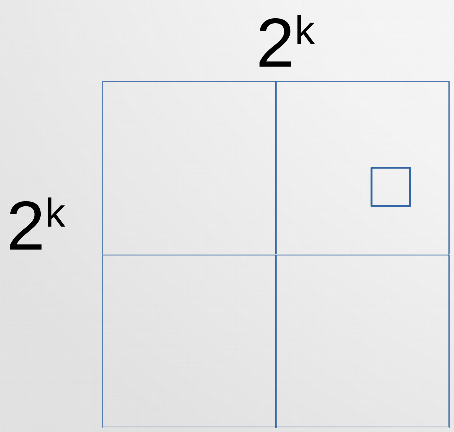


Inductive step: assume that for any arbitrary fixed k, every $2^k \times 2^k$ checkerboard with one square removed can be tiled using right triominoes. (IH)



We need to show that in this case a checkerboard $2^k \times 2^k$ checkerboard with one square removed can be tiles using right triominoes as well.

Section 5.1 Mathematical Induction (continues)

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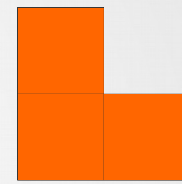
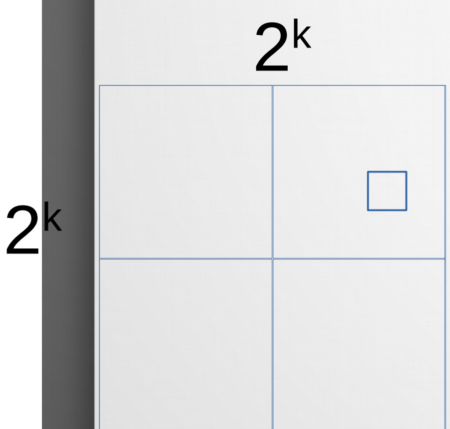
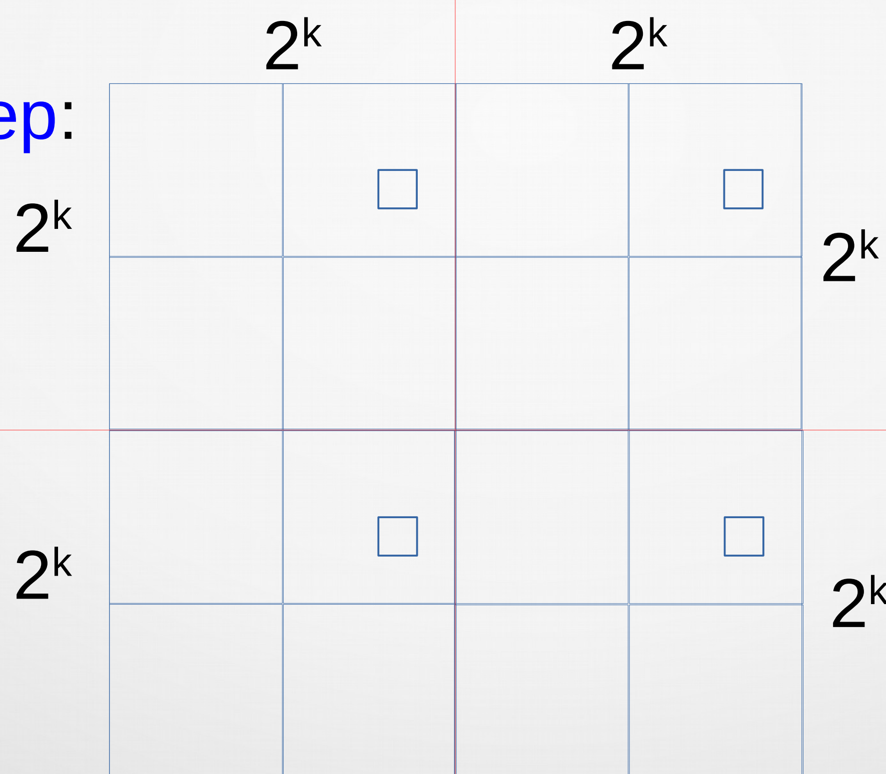


Inductive step:



$2^k+2^k=2^{k+1}$



Cut the $2^{k+1} \times 2^{k+1}$ checkerboard into four checkerboards (each $2^k \times 2^k$). For each of them IH works.

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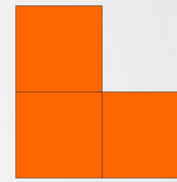
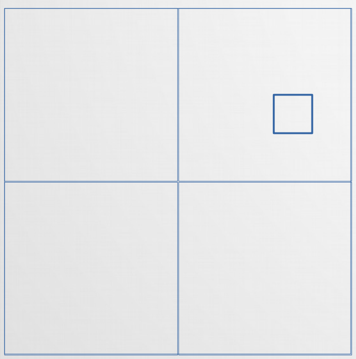


Inductive step:

2^k

2^k

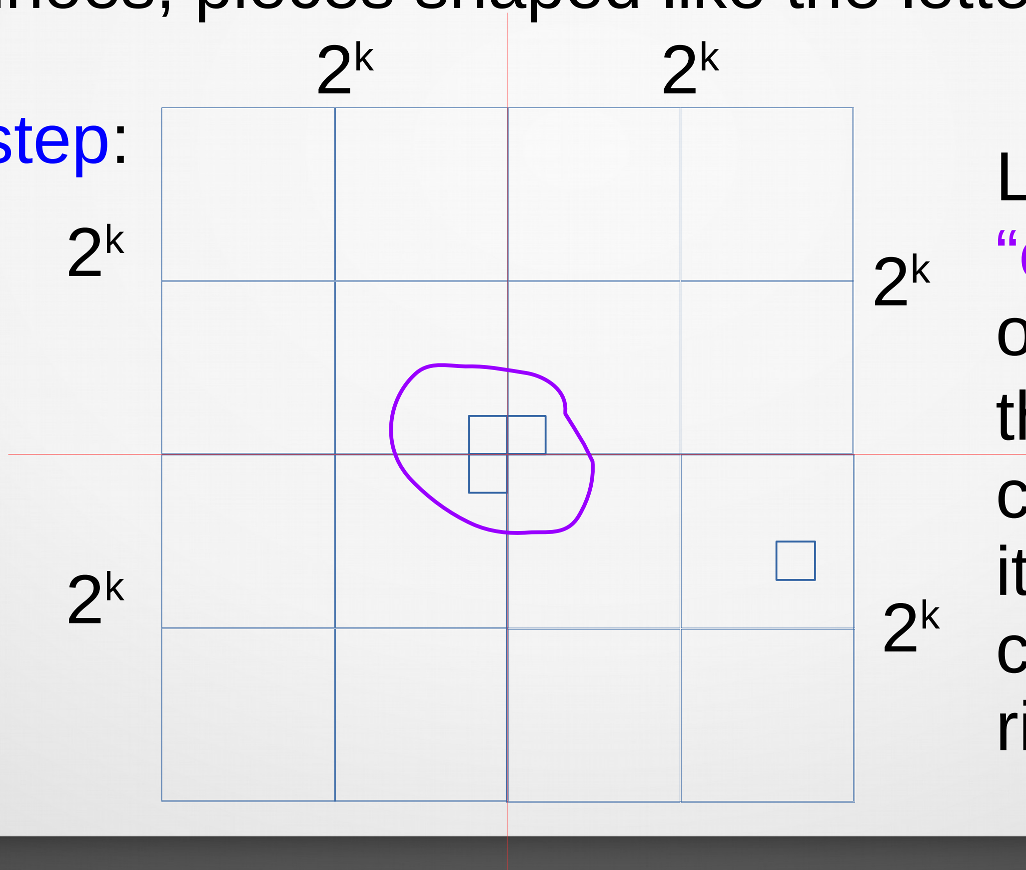


$2^k+2^k=2^{k+1}$

2^k 2^k

2^k 2^k

2^k 2^k



Let's remove the “center” square of each of any three $2^k \times 2^k$ checkerboards – it will be tiled/ covered by the right tiominoe

Section 5.1 Mathematical Induction (continues)

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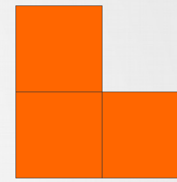
Let n be a positive integer. Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes, pieces shaped like the letter “L.”



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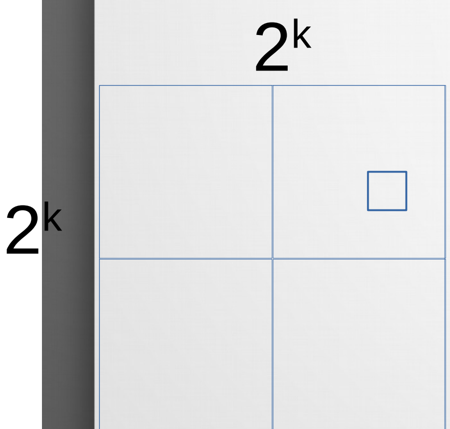

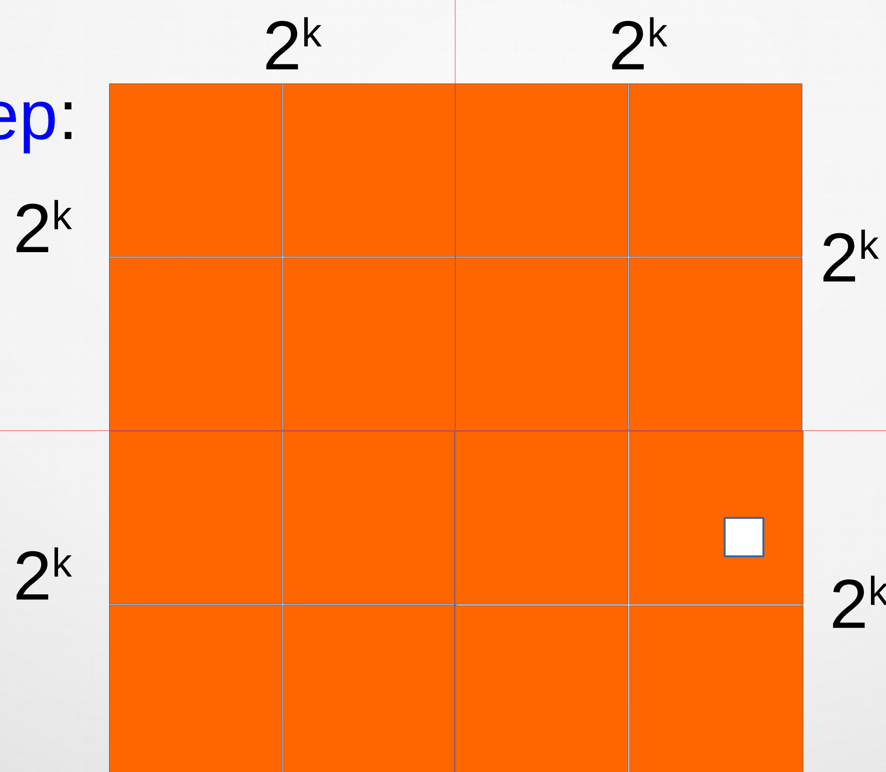

$2^k+2^k=2^{k+1}$

Let's remove the “center” square of each of any three $2^k \times 2^k$ checkerboards – it will be tiled/ covered by the right triominoe.

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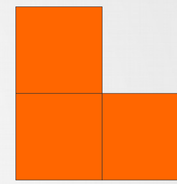

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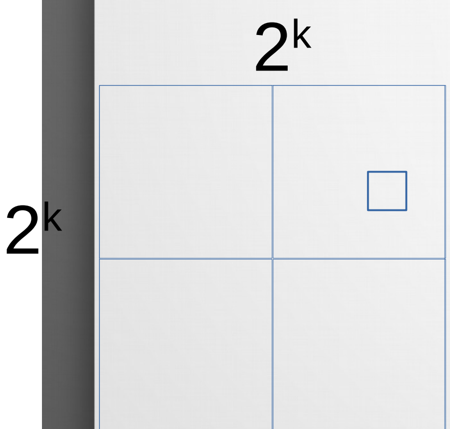

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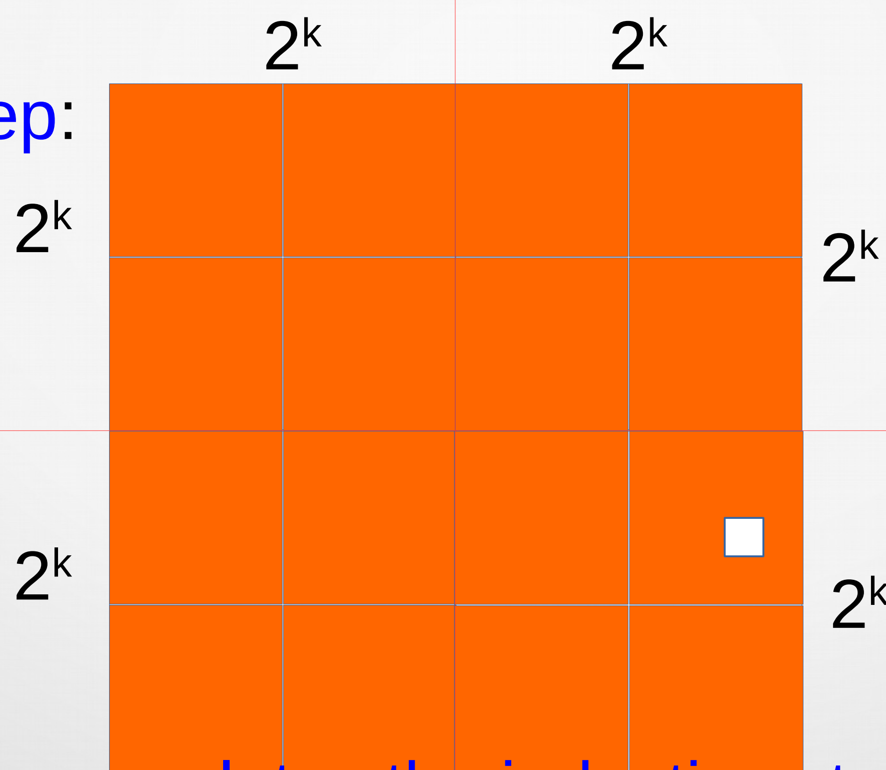


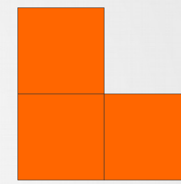
Let's remove the "center" square of each of any three $2^k \times 2^k$ checkerboards – it will be tiled/ covered by the right triominoe.

This completes the inductive step.

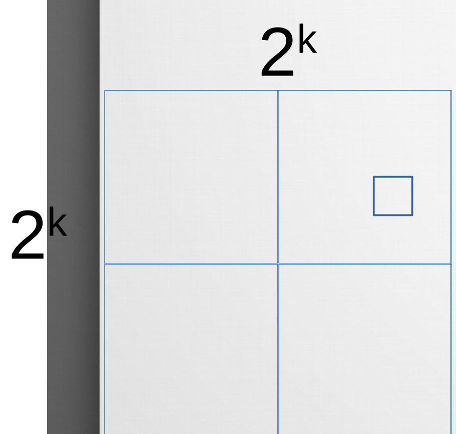
Section 5.1 Mathematical Induction (continues)

Example:

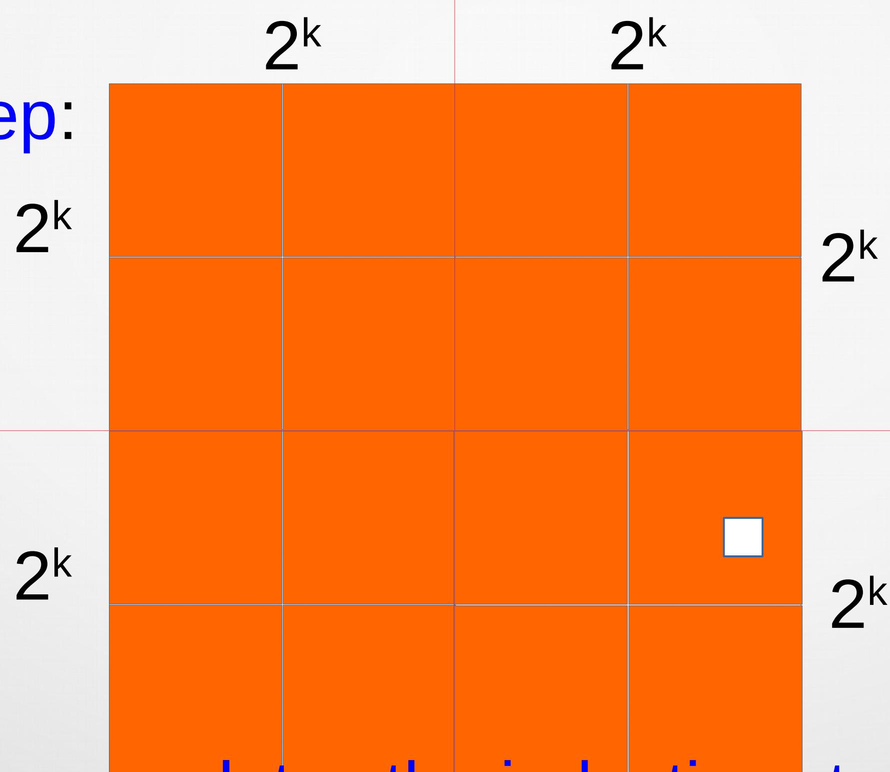
Let n be a positive integer. Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes, pieces shaped like the letter "L."



Inductive step:



$2^k+2^k=2^{k+1}$



Let's remove the "center" square of each of any three $2^k \times 2^k$ checkerboards – it will be tiled/ covered by the right triominoe.

This completes the inductive step.

By mathematical induction we proved P(n) for any positive n.

Section 5.1 Mathematical Induction (continues)

Mistakes in math. induction proofs:

- can be hidden in the basis step
- can be in the inductive step

Section 5.1 Mathematical Induction (continues)

Example: Find what is wrong in the proof of

"Theorem" For every n ∈ **Z**$^+$ $\sum_{i=1}^{n} i = \frac{\left(n+\frac{1}{2}\right)^2}{2}$

Basis step: the formula is true for n = 1

Ind. step: assume for any arb. fixed k ∈ **Z**$^+$, $\sum_{i=1}^{k} i = \frac{\left(k+\frac{1}{2}\right)^2}{2}$

Then $\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{\left(k+\frac{1}{2}\right)^2}{2} + (k+1) = \frac{k^2+k+\frac{1}{4}}{2} + (k+1) = \ldots$

by IH

$$\ldots = \frac{k^2+k+\frac{1}{4}}{2} + \frac{2(k+1)}{2} = \frac{k^2+k+\frac{1}{4}+2k+2}{2} = \frac{k^2+3k+\frac{9}{4}}{2} = \ldots$$

$$\ldots = \frac{\left(k+\frac{3}{2}\right)^2}{2} = \frac{\left((k+1)+\frac{1}{2}\right)^2}{2}$$

As we can see, P(k+1) is also true. This completes the inductive step, and the proof. qed

Section 5.1 Mathematical Induction (continues)

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false $1 != \frac{\left(1+\frac{1}{2}\right)^2}{2} = \frac{9}{8}$

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Section 5.1 Mathematical Induction (continues)

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"Theorem" For every $n \in \mathbf{Z}^+$ $\displaystyle\sum_{i=1}^{n} i = \frac{\left(n+\frac{1}{2}\right)^2}{2}$

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false

$$1! = \frac{\left(1+\frac{1}{2}\right)^2}{2} = \frac{9}{8}$$

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Then
$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{\left(k+\frac{1}{2}\right)^2}{2} + (k+1) = \frac{k^2+k+\frac{1}{4}}{2} + (k+1) = \ldots$$

by IH

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As we can see, P(k+1) is also true.
This completes the inductive step, and the proof. qed

Section 5.2 Strong Induction and well-ordering

Why do we need strong induction?

The Fibonacci sequence is defined by initial values $f_0 = 0$, $f_1 = 1$ and the recurrence relation for $n \geq 2$:

$$f_n = f_{n-1} + f_{n-2}$$

There is an upper bound on the Fibonacci numbers: $f_n \leq 2^n$, for all $n \geq 0$.

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Let's try to prove it using math. Induction:

Basis step: $n = 0$, $f_0 = 0 \leq 2^0 = 1$

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IH

Inductive step: assume for any arbitrary $k \geq 0$, $P(k)$ is true.

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P(k+1): $f_{k+1} = f_k + f_{k-1}$

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$f_n \leq 2^n$, for all $n \geq 0$.

Inductive step:
assume for any arbitrary fixed $k \geq 0$, P(k) is true. IH
P(k+1): $f_{k+1} = f_k + f_{k-1} \leq 2^k + f_{k-1}$, now what to do with f_{k-1}?
(by IH)

Section 5.2 Strong Induction and well-ordering

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$P(k+1)$: $f_{k+1} = f_k + f_{k-1} \leq 2^k + f_{k-1}$, now what to do with f_{k-1}?

(by IH)

The inductive hypothesis for standard induction only provides the bound on f_k. We cannot finish the proof!

Section 5.2 Strong Induction and well-ordering

The principle of strong induction assumes that the fact to be proven holds for *all values less than or equal to* k and proves that the fact holds for k+1.

The standard form of induction only assumes that the fact *holds for* k *in proving that it holds for* k+1.

For the Fibonacci numbers example we can do:

Base case: P(0) and P(1) are true.

Inductive step: we need to show that for every $k \geq 1$,

$(P(0) \wedge P(1) \wedge .... \wedge P(k)) \rightarrow P(k + 1)$.

Section 5.2 Strong Induction and well-ordering

Strong induction:

To prove $P(n)$ is true for all integers $n \geq b$, where $P(n)$ is a propositional function, we complete two steps:

Basis step: verify $P(b)$ is true

Inductive step: show that for every $k \geq b$,

$$(P(b) \wedge P(b+1) \wedge .... \wedge P(k)) \rightarrow P(k+1) \text{ is true}$$

i.e. we assume that for every $k \geq b$, $P(b) \wedge P(b+1) \wedge .... \wedge P(k)$ is true and show that in this case $P(k+1)$ is also true.

Section 5.2 Strong Induction and well-ordering

Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$

Example: Let's prove the upper bound on the Fibonacci numbers: $f_n \leq 2^n$, for all $n \geq 0$.

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Basis step: $n = 0$, $f_0 = 0 \leq 2^0 = 1$

$n = 1$ $f_1 = 1 \leq 2^1 = 2$

Inductive step: assume for any arbitrary $k \geq 0$, IH

$P(0) \wedge P(1) \wedge P(2) \wedge \ldots \wedge P(k)$ is true

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$n = 1$ $f_1 = 1 \leq 2^1 = 2$

Inductive step: assume for any arbitrary $k \geq 0$, IH

$P(0) \wedge P(1) \wedge P(2) \wedge \ldots \wedge P(k)$ is true, i.e.

$$\forall j \left(j \in Z \wedge 0 \leq j \leq k \right) P(j)$$

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Let's show that is this case, P(k+1) is also true:

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$f_{k+1} = f_k + f_{k-1} \leq 2^k + 2^{k-1} =$

by IH twice

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Let's show that is this case, $P(k+1)$ is also true:

$f_{k+1} = f_k + f_{k-1} \leq 2^k + 2^{k-1} = 2^{k-1}(2+1) = 3 \cdot 2^{k-1}$

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by IH twice

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by IH twice

We showed that $P(k+1)$ is also true

This completes the inductive step.

Section 5.2 Strong Induction and well-ordering

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$n = 1$ $f_1 = 1 \leq 2^1 = 2$

Inductive step: assume for any arbitrary $k \geq 0$, IH

$P(0) \wedge P(1) \wedge P(2) \wedge \ldots \wedge P(k)$ is true.

We showed that P(k+1) is also true

By strong induction we proved that $f_n \leq 2^n$, for all $n \geq 0$.

q.e.d.

Section 5.2 Strong Induction and well-ordering

Multiple base cases:

As you noticed, we had two base cases in the basis step.

Basis step is not limited to one base case.

To some extent, determining how many base cases are required in a strong induction proof requires some trial and error, but some common sense reasoning can be applied

Section 5.2 Strong Induction and well-ordering

<u>Example (multiple base cases) (from ZyBooks)</u>:

Cans of juice come in packs of 3 or 4. For any $n \geq 6$ it is possible to buy n cans of juice by purchasing a combination of 3-packs or 4-packs only.

Section 5.2 Strong Induction and well-ordering

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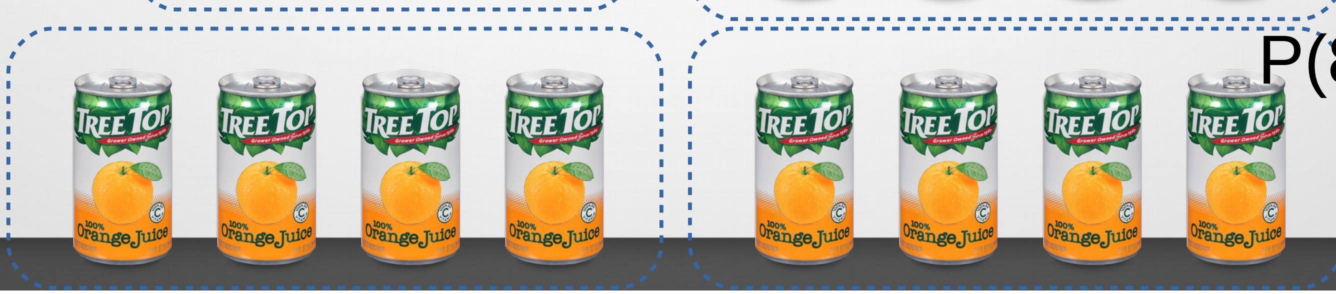
Basis step:

P(6) is true

P(7) is true



P(8) is true

Section 5.2 Strong Induction and well-ordering

Example (multiple base cases) (from ZyBooks):

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Inductive step: assume that for any $k \geq 8$,

$P(6) \wedge \ldots \wedge P(k)$ is true.

Let's show that in this case $P(k+1)$ is true:

Section 5.2 Strong Induction and well-ordering

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Let's show that in this case $P(k+1)$ is true:

Since $P(k-2)$ is true, i.e. $k-2$ cans can be presented as a combination of 3-packs and 4-packs, then by adding one 3-pack it will be getting exactly $k+1$ cans in total.

$(k-2) + 3 = k+1$

Section 5.2 Strong Induction and well-ordering

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$(k-2) + 3 = k+1$
Hence $P(k+1)$ is also true.
This completes the inductive step.

Section 5.2 Strong Induction and well-ordering

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Cans of juice come in packs of 3 or 4. For any $n \geq 6$ it is possible to buy n cans of juice by purchasing a combination of 3-packs or 4-packs only.

Inductive step: assume that for any $k \geq 8$,

$$P(6) \wedge \ldots \wedge P(k) \text{ is true.}$$

Let's show that in this case $P(k+1)$ is true:

Since $P(k-2)$ is true, i.e. $k-2$ cans can be presented as a combination of 3-packs and 4-packs, then by adding one 3-pack it will be getting exactly $k+1$ cans in total.

$(k-2) + 3 = k+1$

Hence $P(k+1)$ is also true.

This completes the inductive step.

By strong induction the statement above is true.

q.e.d.

Section 5.2 Strong Induction and well-ordering

Well-ordering:

The **well-ordering principle** says that any non-empty subset of the non-negative integers has a smallest element. (Axiom 5, page A-5)

We used it before to show that mathematical induction is valid.

It is possible to use this principle directly in proofs.

Furthermore, the well-ordering principle, the mathematical induction and strong induction are all equivalent.

Section 5.2 Strong Induction and well-ordering

Example: use the well-ordering principle to prove the division algorithm.

If $a \in Z$, and $d \in Z^+$, then $\exists!\ q \in Z$ and $\exists!\ r \in Z$ $(0 \le r < d)$ such that $a = dq+r$.

$$17 = 5*3 + 2$$

divisor

remainder

a (dividend)

quotient

Section 5.2 Strong Induction and well-ordering

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Proof: Assume $a \in Z$, and $d \in Z^+$.

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Proof: Assume $a \in Z$, and $d \in Z^+$.

Let S be the set of non-negative integers of the form $a - dq$, where $q \in Z$.

Section 5.2 Strong Induction and well-ordering

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Proof: Assume $a \in Z$, and $d \in Z^+$.

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$|S| \neq 0$ because $-dq$ can be made as large as desired (with q as negative integer and large absolute value)

Section 5.2 Strong Induction and well-ordering

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Therefore, $0 \leq r < d$

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d divides $r-r_1$. In addition, $-d < r-r_1 < d$.

Therefore, $r=r_1$, and hence $q_0 = q_1$. We got a contradiction.

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By the well-ordering principle, S has a smallest element

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And we proved that q_0 and r are unique!

This completes the proof of the division algorithm.

q.e.d.