Section 5.1 Mathematical Induction

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Can be proven with mathematical induction

Section 5.1 Mathematical Induction

Principle of mathematical induction:

Assume that $P(n)$ is a propositional function. To prove that $P(n)$ is true for all positive integers n we complete two steps:

BASIS STEP (BASE): We verify that $P(1)$ is true

note: it is not always 1

INDUCTIVE STEP: We show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

to do the inductive step: we assume that $P(k)$ is true for an arbitrary positive k and show that under this assumption $P(k+1)$ must also be true.

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inductive hypothesis

Review: Functions

Let $f(x) = 2x+3$

What is $f(x+1)$?

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Let $g(x) = 3x - 4x^2 + 19$
What is $g(x-1)$?

$$\begin{aligned} g(x-1) &= 3(x-1) - 4(x-1)^2 + 19 = \\ &= 3x - 3 - 4(\,x^2 - 2x+1) + 19 = \\ &= 3x - 3 - 4x^2 + 8x - 4 + 19 = \\ &= -4x^2 + 11x + 12 \end{aligned}$$

Section 5.1 Mathematical Induction

Example: Let's prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, i.e.
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We need to show that P(k+1) is true

$$P(k+1): 1+2+3+\ldots+k+1=\frac{(k+1)((k+1)+1)}{2}=\frac{(k+1)(k+2)}{2}$$

Video link: https://www.khanacademy.org/math/algebra-home/alg-series-and-induction/alg-induction/v/proof-by-induction

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$$P(k): 1+2+3+\ldots+k=\frac{k(k+1)}{2}$$

$$1+2+\ldots+k+(k+1)=\ldots$$

pulling out last term

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We showed that if P(k) holds then P(k+1) holds.
This completes the inductive step.

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$$P(k): 1+2+3+\ldots+k=\frac{k(k+1)}{2}$$

By math. induction, P(n) is true for any positive integer n.

q.e.d (Quod Erat Demonstrandum)

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that which was to be demonstrated

Section 5.1 Mathematical Induction

Visualizations of mathematical induction:

climbing an infinite ladder

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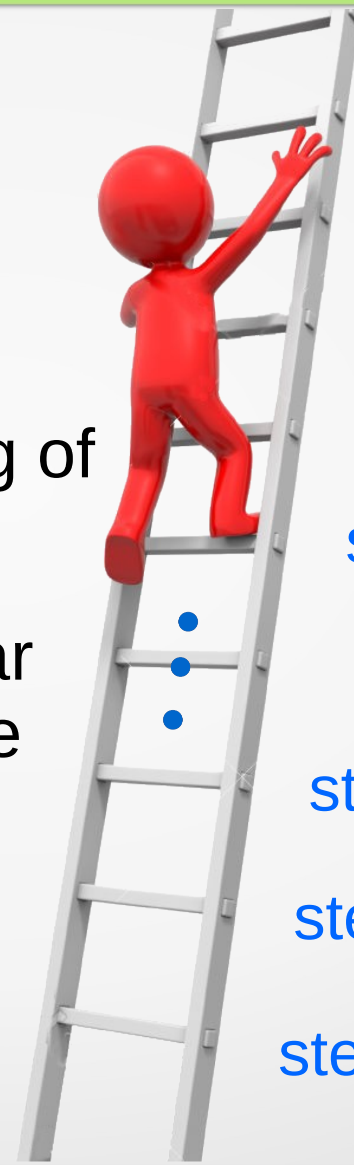
<u>Visualizations of mathematical induction</u>:

climbing an infinite ladder

- we can reach the first rung of the ladder $P(1)$
- If we can reach a particular rung of the ladder, then we can reach the next rung

$P(k) \rightarrow P(k+1)$

Base: we can reach step 1



$P(n)$ – we can reach n^{th} rung

step k+1

step k

step 3

step 2

step 1

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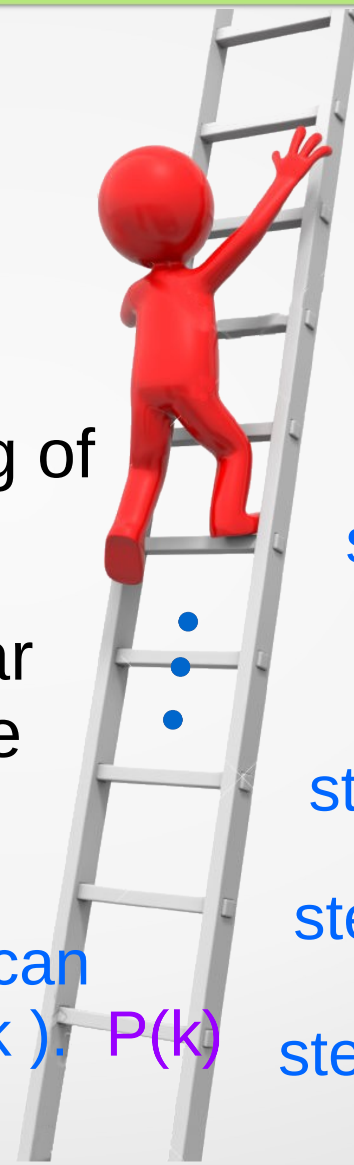

$P(n)$ – we can reach n^{th} rung

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Induction step: assume that we can reach k^{th} rung (for any arbitrary k). $P(k)$

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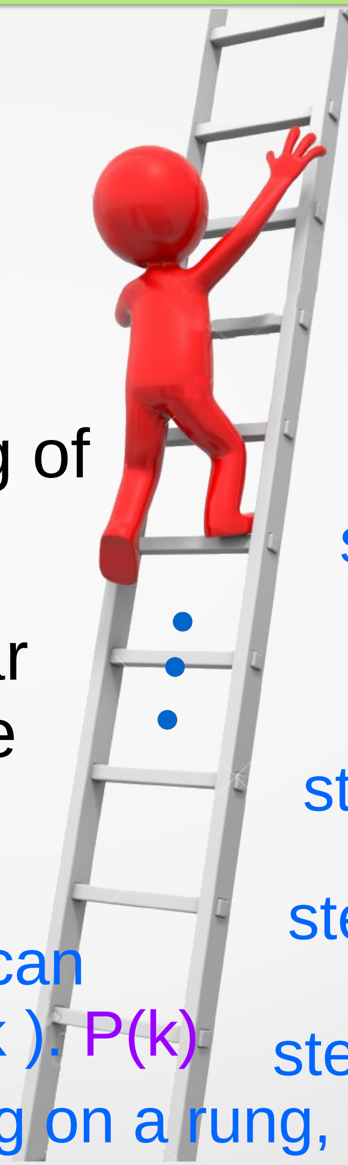

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$P(k) \rightarrow P(k+1)$

step k+1

step k

step 3

step 2

step 1

Induction step: assume that we can reach k^{th} rung (for any arbitrary k): $P(k)$

We also know that when standing on a rung, we can reach next rung.

Section 5.1 Mathematical Induction

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climbing an infinite ladder

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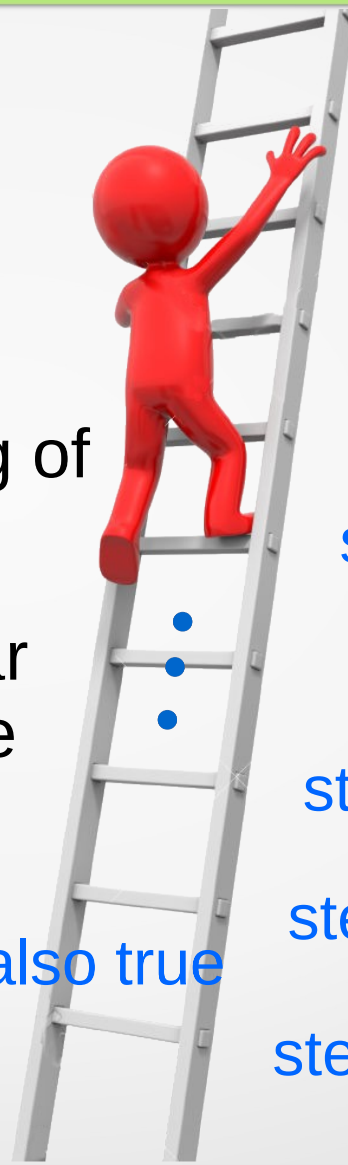
- If we can reach a particular rung of the ladder, then we can reach the next rung

$P(k) \rightarrow P(k+1)$

Induction step: hence P(k+1) is also true



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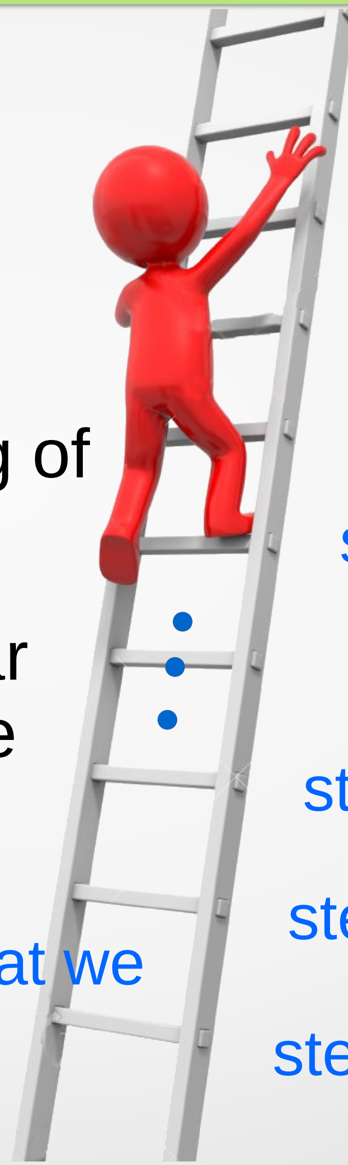
- If we can reach a particular rung of the ladder, then we can reach the next rung

$P(k) \rightarrow P(k+1)$

By math. induction we proved that we can climb an infinite ladder

qed



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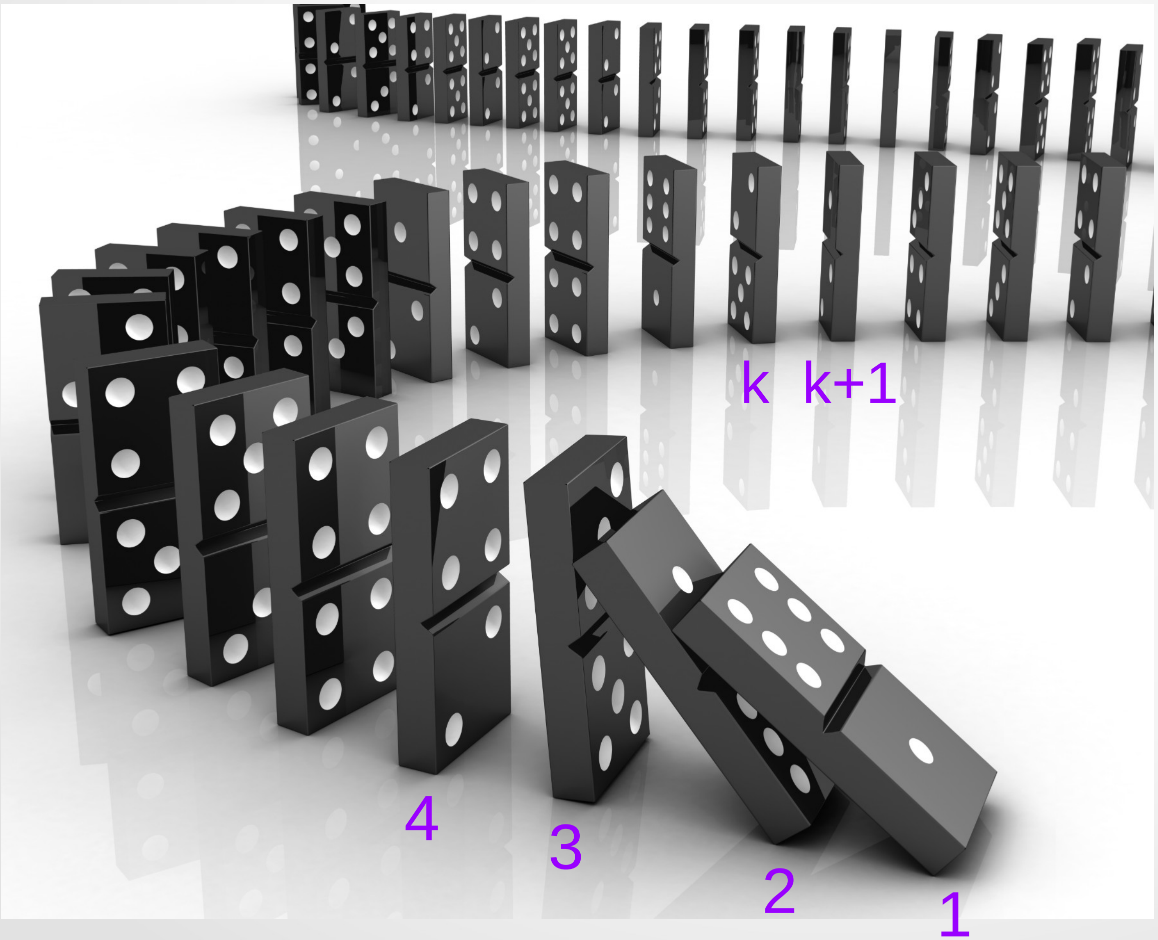

$P(n)$ – domino n is knocked over

we can knock over the 1st domino $P(1)$

If kth domino is knocked it knocks over the next domino (k+1)th
$P(k) \rightarrow P(k+1)$

Section 5.1 Mathematical Induction

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Proof (by contradiction): assume that $P(1)$ it true, and $P(k) \to P(k+1)$ is true for all positive integers k.
We need to show that in this case $P(n)$ is true for all positive integers n.

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Assume it is false.

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Assume it is false. i.e. math induction doesn't work (proof by contradiction).

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We need to show that in this case $P(n)$ is true for all positive integers n.
Assume it is false. i.e. math induction doesn't work (proof by contradiction).
In this case there is at least one positive integer i for which $P(i)$ is false.
Let S be the set of positive integers for which $P(n)$ is false.

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Proof (by contradiction): $P(1)$ it true and $\forall k \in Z^+ \, (P(k) \to P(k+1))$
...
$\exists\, i \in Z^+$ ($P(i)$ is false)
Let S be the set of positive integers for which $P(n)$ is false.
Set S has a least element according to The Well-Ordering Property, let's name it m. $P(m)$ is false.

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$m \neq 1$ because $P(1)$ it true, hence $m > 1$

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$m-1$ is a positive integer, therefore $P(m-1)$ must be true

(m is the smallest where $P(n)$ fails).

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We need to show that in this case $P(n)$ is true for all positive integers n.
Our assumption was false!
~~Assume it is false.~~ i.e. math induction doesn't work (proof by contradiction)

So there is no positive integer at which $P(n)$ fails.
Therefore $P(n)$ is true for all positive integers n.

q.e.d. (Quod Erat Demonstrandum)

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Section 5.1 Mathematical Induction

<u>The good and the bad of mathematical induction</u>

<u>good</u>: we can prove a conjecture (statement) once it is made and is true.

<u>bad</u>: math. induction cannot be used to find new theorems

proofs by math. induction do not provide insights as to why theorems are true

Section 5.1 Mathematical Induction

Example 1: Let P(n) be the statement that

$$1^3+2^3+3^3+\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

for any positive integer *n*.

(*Rosen, p. 329 № 4*)

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Proof:

1) Base step: $P(1): 1^3=1=\left(\frac{1(1+1)}{2}\right)^2=1$, hence the statement is true for *n=1*

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the IH (Inductive Hypotheses): assume that P(k) is true for an arbitrary fixed integer k ≥ 1

$$1^3+2^3+3^3+\ldots+k^3=\left(\frac{k(k+1)}{2}\right)^2$$

Let's prove that in this case P(k+1) is also true.

Section 5.1 Mathematical Induction

Example 1: Let P(n) be the statement that

$$1^3+2^3+3^3+\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

Proof:

2) Inductive step:

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Let's prove that P(k+1) is also true:

$$1^3+2^3+3^3+\ldots+k^3+(k+1)^3=$$

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This completes the inductive step

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(*Rosen, p. 329 № 4*) for any positive integer *n*.

Therefore, by mathematical induction P(n) is true for all positive integers n.

qed