Chapter 5 Sequences and Summations

Sequences:

A sequence is a special type of function in which the domain is a consecutive set of integers.

4, 7, 3, 9, 4

a_1 a_2 a_3 a_4 a_5

Chapter 5 Sequences and Summations

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$a_1\ a_2\ a_3\ a_4\ a_5$ ← terms of the sequence

indices of the sequence

Chapter 5 Sequences and Summations

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a_1 a_2 a_3 a_4 a_5 ← terms of the sequence

initial index   final index

Chapter 5 Sequences and Summations

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4, 7, 3, 9, 4

$a_1\ a_2\ a_3\ a_4\ a_5 \longleftarrow$ terms of the sequence

initial index final index

$f(1) = 8,\ f(2) = 16,\ f(3) = 24,\ f(4) = 32 \qquad f(k) = 8k$

Chapter 5 Sequences and Summations

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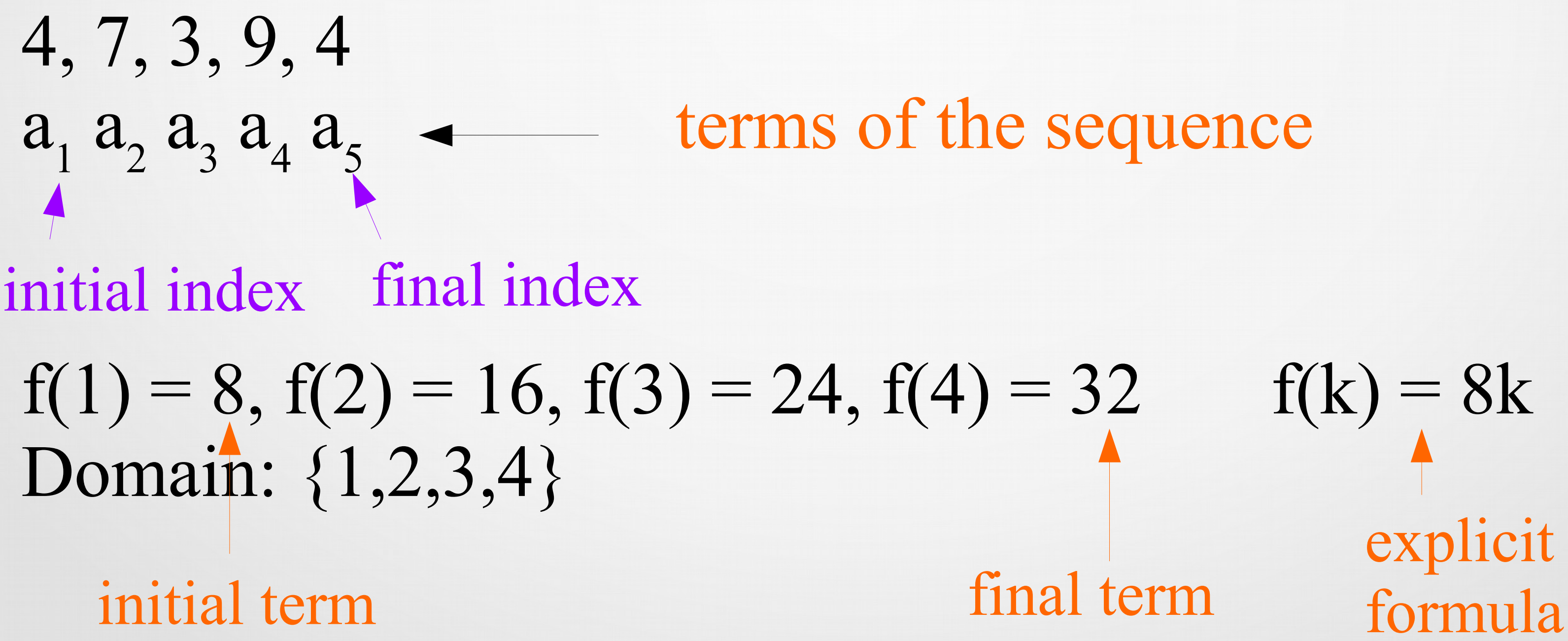
f(1) = 8, f(2) = 16, f(3) = 24, f(4) = 32 f(k) = 8k

Domain: {1,2,3,4}

Chapter 5 Sequences and Summations

Sequences:

A sequence is a special type of function in which the domain is a consecutive set of integers.



4, 7, 3, 9, 4

$a_1\ a_2\ a_3\ a_4\ a_5 \longleftarrow$ terms of the sequence

initial index final index

f(1) = 8, f(2) = 16, f(3) = 24, f(4) = 32 f(k) = 8k

Domain: {1,2,3,4}

initial term final term explicit formula

Chapter 5 Sequences and Summations

Sequences:

A sequence is a special type of function in which the domain is a consecutive set of integers.

4, 7, 3, 9, 4

$a_1\ a_2\ a_3\ a_4\ a_5$

Finite sequences

$f(1) = 8, f(2) = 16, f(3) = 24, f(4) = 32$

Chapter 5 Sequences and Summations

Sequences:

A sequence is a special type of function in which the domain is a consecutive set of integers.

1, 3, 5, 7, 9, 11, 13, ...

$$\frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, -\frac{1}{7}, \ldots$$

20, 14, 8, 2, -4, -10, ...

infinite sequences

Chapter 5 Sequences and Summations

Increasing and decreasing sequences:

A sequence is increasing if for every two consecutive indices, k and k + 1, in the domain, $a_k < a_{k+1}$.

1, 5, 6, 8, 9, 12, 19

A sequence is non-decreasing if for every two consecutive indices, k and k + 1, in the domain, $a_k \leq a_{k+1}$.

1, 1, 5, 6, 8, 8, 12, 19

Chapter 5 Sequences and Summations

Increasing and decreasing sequences:

A sequence is increasing if for every two consecutive indices, k and k + 1, in the domain, $a_k < a_{k+1}$.

strictly increasing

1, 1, 5, 6, 8, 8, 12, 19

A sequence is non-decreasing if for every two consecutive indices, k and k + 1, in the domain, $a_k \leq a_{k+1}$.

increasing

1, 5, 6, 8, 9, 12, 19

Note that an increasing sequences is non-decreasing as well

Chapter 5 Sequences and Summations

Increasing and decreasing sequences:

A sequence is decreasing if for every two consecutive indices, k and k + 1, in the domain, $a_k > a_{k+1}$.

23, 18, 15, 5, 2, -3

A sequence is non-increasing if for every two consecutive indices, k and k + 1, in the domain, $a_k \geq a_{k+1}$.

24, 23, 14, 14, 14, 7, 6

Chapter 5 Sequences and Summations

Increasing and decreasing sequences:

A sequence is decreasing if for every two consecutive indices, k and k + 1, in the domain, $a_k > a_{k+1}$.

strictly decreasing

23, 18, 15, 5, 2, -3

A sequence is non-increasing if for every two consecutive indices, k and k + 1, in the domain, $a_k \geq a_{k+1}$.

decreasing

24, 23, 14, 14, 14, 7, 6

Note that an decreasing sequences is non-increasing as well

Chapter 5 Sequences and Summations

Arithmetic sequence:

An arithmetic sequence is a sequence of real numbers where each term after the initial term is found by taking the previous term and adding a fixed number called the *common difference*. An arithmetic sequence can be finite or infinite.

Example: -8, -4, 0, 4, 8, 12, …

Chapter 5 Sequences and Summations

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initial value $a_0 = -8$ *common difference* $= 4$

Chapter 5 Sequences and Summations

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Example: -8, -4, 0, 4, 8, 12, …

initial value $a_0 = -8$ *common difference* = 4

We can find the nth element of the sequence by the explicit formula defining the sequence:
$a_n = a_0 + dn$, for $n \geq 0$, where d is the *common difference*

Chapter 5 Sequences and Summations

Geometric sequence:

A geometric sequence is a sequence of real numbers where each term after the initial term is found by taking the previous term and multiplying by a fixed number called the *common ratio*. A geometric sequence can be finite or infinite.

Example: 3, -6, 12, -24, 48, -96, …

Chapter 5 Sequences and Summations

Geometric sequence:

A geometric sequence is a sequence of real numbers where each term after the initial term is found by taking the previous term and multiplying by a fixed number called the *common ratio*. A geometric sequence can be finite or infinite.

Example: 3, -6, 12, -24, 48, -96, …

initial value $t_0 = 3$ *common ratio* = -2

Chapter 5 Sequences and Summations

Geometric sequence:

A geometric sequence is a sequence of real numbers where each term after the initial term is found by taking the previous term and multiplying by a fixed number called the *common ratio*. A geometric sequence can be finite or infinite.

Example: 3, -6, 12, -24, 48, -96, …

initial value $t_0 = 3$ *common ratio* = -2

We can find the nth element of the sequence by the explicit formula defining the sequence:

$t_n = t_0 \times r^n$, for $n \geq 0$, where r is the *common ratio*

Chapter 5 Sequences and Summations

Summations:

Summation notation is used to express the sum of terms in a numerical sequence. Consider a sequence:

upper limit

3, -6, 1, -2, 8, -9

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6$

$$\sum_{i=1}^{6} a_i = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 3 + (-6) + 1 + (-2) + 8 + (-9) = -5$$

lower limit

index of the summation

Chapter 5 Sequences and Summations

Summations:

Summation notation is used to express the sum of terms in a numerical sequence. Consider a general finite sequence:

upper limit

$$a_s \quad a_{s+1} \quad a_{s+2} \quad \ldots \quad a_k$$

$$\sum_{i=s}^{k} a_i = a_s + a_{s+1} + a_{s+2} + \ldots + a_k$$

lower limit

index of the summation

Chapter 5 Sequences and Summations

Summations:

Example: Find $\sum_{j=-2}^{3} j^2$

Solution:

$$\sum_{j=-2}^{3} j^2=(-2)^2+(-1)^2+0^2+1^2+2^2+3^2=4+1+0+1+4+9=19$$

summation form

expanded form

Answer: $\sum_{j=-2}^{3} j^2=19$

Chapter 5 Sequences and Summations

Pulling out the last term of the summation:

Often it is useful to be able to pull out the last term of the summation.

Example:

$$\sum_{j=-2}^{3} j^2 = (-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2 + 3^2$$

$$\sum_{j=-2}^{3} j^2 = \sum_{j=-2}^{2} j^2 + 3^2$$

Chapter 5 Sequences and Summations

Pulling out the last term of the summation:

Often it is useful to be able to pull out the last term of the summation.

Example 1: $\sum_{k=m}^{n} a_k = a_m + \ldots + a_{n-1} + a_n = \sum_{k=m}^{n-1} a_k + a_n$

Chapter 5 Sequences and Summations

Pulling out the last term of the summation:

Often it is useful to be able to pull out the last term of the summation.

Example 1: $$\sum_{k=m}^{n} a_k = a_m + \ldots + a_{n-1} + a_n = \sum_{k=m}^{n-1} a_k + a_n$$

Example 2: $$\sum_{k=3}^{n} (k+2)^4 = \sum_{k=3}^{n-1} (k+2)^4 + (n+2)^4$$

Chapter 5 Sequences and Summations

Infinite summations:

$$\sum_{i=2}^{\infty} i^3 = 2^3 + 3^3 + 4^3 + 5^3 + \ldots$$

Chapter 5 Sequences and Summations

properties of summations:

$$\sum_{i=1}^{k} c = \underbrace{c+c+c+\ldots+c}_{k \text{ of } c\text{s}} = k\,c$$

, where c is some constant

Chapter 5 Sequences and Summations

properties of summations:

$$\sum_{i=1}^{k} c = \underbrace{c + c + c + \ldots + c}_{k \text{ of } c\text{s}} = k\,c$$, where c is some constant

Examples:

$$\sum_{i=1}^{78} 10 = 78 \times 10 = 780$$

Chapter 5 Sequences and Summations

properties of summations:

$$\sum_{i=1}^{k} c = \underbrace{c+c+c+\ldots+c}_{k \text{ of } c\text{s}} = k\,c$$, where c is some constant

Examples:

$$\sum_{i=1}^{78} 10 = 78 \times 10 = 780$$

$$\sum_{i=11}^{123} 5 = (123-11+1) \times 5 = 113 \times 5 = 565$$

Chapter 5 Sequences and Summations

properties of summations:

$$\sum_{i=1}^{k} c = \underbrace{c+c+c+\ldots+c}_{k \text{ of } c\text{s}} = k\,c$$

, where c is some constant

first term

last term

$$\sum_{i=1}^{n} i = 1+2+3+\ldots+n = \frac{(1+n)n}{2}$$

number of terms

needs to be proved!

Chapter 5 Sequences and Summations

properties of summations:

$$\sum_{i=1}^{k} c = \underbrace{c+c+c+\ldots+c}_{k \text{ of } c\text{s}} = k\,c$$

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last term

$$\sum_{i=1}^{n} i = 1+2+3+\ldots+n = \frac{(1+n)n}{2}$$

number of terms

needs to be proved!

Examples:

number of terms

$$\sum_{i=1}^{90} i = 1+2+3+\ldots+90 = \frac{(1+90)\overset{\downarrow}{90}}{2} = 4095$$

Chapter 5 Sequences and Summations

properties of summations:

$$\sum_{i=1}^{k} c = \underbrace{c+c+c+\ldots+c}_{k \text{ of } c\text{s}} = k\,c$$

, where c is some constant

first term

last term

$$\sum_{i=1}^{n} i = 1+2+3+\ldots+n = \frac{(1+n)n}{2}$$

number of terms

needs to be proved!

Examples:

first term

number of terms

$$\sum_{i=11}^{76} i = 11+12+13+\ldots+76 = \frac{(11+76)(76-11+1)}{2} = 2871$$

Chapter 5 Sequences and Summations

properties of summations:

$$\sum_{i=1}^{k} c = \underbrace{c+c+c+\ldots+c}_{k \text{ of } c\text{s}} = k\,c$$

, where c is some constant

first term

last term

$$\sum_{i=1}^{n} i = 1+2+3+\ldots+n = \frac{(1+n)n}{2}$$

number of terms

needs to be proved!

$$\sum_{i=1}^{n} (c+di) = (c+d)+(c+2d)+\ldots+(c+nd) = cn + d\sum_{i=1}^{n} i = \ldots$$

, where c and d are some constants

Chapter 5 Sequences and Summations

properties of summations:

$$\sum_{i=1}^{k} c = \underbrace{c+c+c+\ldots+c}_{k \text{ of } c\text{s}} = k\,c$$

, where c is some constant

first term

last term

$$\sum_{i=1}^{n} i = 1+2+3+\ldots+n = \frac{(1+n)n}{2}$$

number of terms

needs to be proved!

$$\sum_{i=1}^{n} (c+di) = (c+d)+(c+2d)+\ldots+(c+nd) = cn+d\sum_{i=1}^{n} i = cn+d\,\frac{(1+n)n}{2}$$

arithmetic sequence?
just change the limits...

, where c and d are some constants

Chapter 5 Sequences and Summations

properties of summations:

$$\sum_{i=1}^{k} c = \underbrace{c+c+c+\ldots+c}_{k \text{ of } c\text{s}} = k\,c$$

, where c is some constant

first term

last term

$$\sum_{i=1}^{n} i = 1+2+3+\ldots+n = \frac{(1+n)n}{2}$$

number of terms

needs to be proved!

arithmetic sequence

$a_0 = c,\ a_1 = c+d,\ a_2 = c+2d, \ldots$

$$\sum_{i=0}^{n-1} (c+di) = c+(c+d)+\ldots+(c+d(n-1)) = cn+d\sum_{i=0}^{n-1} i = \ldots$$

$$\ldots = cn+d\frac{(0+n-1)n}{2} = cn+d\frac{n(n-1)}{2}$$

, where c is the *first term*, and d is *common difference*

Chapter 5 Sequences and Summations

properties of summations:

For any arithmetic sequence $\{a_k\}$ with *initial index* 1, the sum of the first n terms (partial sum) can be found by:

$$\sum_{i=1}^{n} a_i = (a_1 + a_n)\frac{n}{2}$$

number of terms

first term last term

needs to be proved!

Chapter 5 Sequences and Summations

properties of summations:

For any arithmetic sequence $\{a_k\}$ with *initial index* 0, the sum of the first n terms (partial sum) can be found by:

$$\sum_{i=0}^{n-1} a_i = (a_0 + a_{n-1})\frac{n}{2}$$

number of terms

first term last term

needs to be proved!

Chapter 5 Sequences and Summations

properties of summations:

For any arithmetic sequence $\{a_k\}$ with *initial index* 0, the sum of the first n terms (partial sum) can be found by:

$$\sum_{i=0}^{n-1} a_i = \left(a_0 + a_{n-1}\right)\frac{n}{2}$$

number of terms

first term last term

needs to be proved!

For any geometric sequence $\{a_k\}$ with *initial index* 0, *initial term c* and *common ratio* $r \neq 1$, the sum of the first n terms (partial sum) can be found by:

$$\sum_{i=0}^{n-1} a_i = \sum_{i=0}^{n-1} cr^i = c + cr + cr^2 + \ldots + cr^{n-1} = \frac{c\left(r^n - 1\right)}{r - 1}$$

needs to be proved!

Chapter 5 Sequences and Summations

Changing the index variables and limits in the summations:

$$\sum_{i=1}^{n} b_i = \sum_{j=1}^{n} b_j$$ simply renaming the index variable

$$\sum_{\substack{i=1\\-1}}^{n\ -1} b_i \underset{+1}{=} \sum_{i=0}^{(n-1)} b_{i+1}$$ changing the limits

$$\sum_{i=1}^{n} i^2 = \sum_{i=0}^{(n-1)} (i+1)^2$$ changing the limits

$$\sum_{i=-1\,+3}^{n+3} (2\overset{-3}{i}+7) = \sum_{i=2}^{n+3} (2(i-3)+7) = \sum_{i=2}^{n+3} (2i+1)$$ changing the limits

Chapter 5 Sequences and Summations

Changing the index variables and limits in the summations:

Example: for the following summation's lower limit

$$\sum_{i=-5}^{10} (3i+i^2)$$

a) from -5 to 0

b) from -5 to 1

Chapter 5 Sequences and Summations

<u>Changing the index variables and limits in the summations:</u>

Example: for the following summation's lower limit $\sum_{i=-5}^{10}(3i+i^2)$

a) from -5 to 0

$$\sum_{i=-5\,+5}^{10\,+5} \overset{-5}{(3i+i^2)} = \sum_{i=0}^{15}\left(3(i-5)+(i-5)^2\right) = \sum_{i=0}^{15}(i^2-7i+10)$$

b) from -5 to 1

Chapter 5 Sequences and Summations

Changing the index variables and limits in the summations:

Example: for the following summation's lower limit $\sum_{i=-5}^{10}(3i+i^2)$

a) from -5 to 0

$$\sum_{i=-5\,+5}^{10\,+5}\overset{-5}{(3i+i^2)}=\sum_{i=0}^{15}(3(i-5)+(i-5)^2)=\sum_{i=0}^{15}(i^2-7i+10)$$

b) from -5 to 1

$$\sum_{i=-5\,+6}^{10\,+6}\overset{-6}{(3i+i^2)}=\sum_{i=1}^{16}(3(i-6)+(i-6)^2)=\sum_{i=1}^{16}(i^2-9i+18)$$

Chapter 5 Sequences and Summations

Changing the index variables and limits in the summations:

Example:

$$\sum_{i=-5}^{10}\left(3i+i^2\right)=\sum_{i=0}^{15}\left(i^2-7i+10\right)=\sum_{i=1}^{16}\left(i^2-9i+18\right)$$