p. 342/11 $P(j)$:

Player 1 wins if $n=4j$, $4j+2$ or $4j+3$

Player 2 wins if $n=4j+1$

$j \in \mathbb{Z}$, $j \geq 0$.

Proof:

Basis case: $j=0$

- $n=0$ not applicable
- $n=0+2=2$, here is the strategy for Player 1 to win:

  Player 1: pick 1 match, then Player 2 will be left with 1 match – he/she lost
- $n=0+3=3$, here is the strategy for Player 1 to win:

  Player 1: pick 2 matches, then Player 2 will be left with 1 match – he/she lost

- $n=1$, in this case Player 1 lost, he/she will have to take this one match.

$j=1$

- $n=4j=4\cdot1=4$, then Player 1 will take 3 matches, and Player 2 is left with one match – he/she lost
- $n=4j+2=4\cdot1+2=6$, then Player 1 will take 1 match, 5 will be left

if Player 2 takes 1 match, then Player 1 will take 3 matches and Player 2 is left with 1 match – he/she lost

if Player 2 takes 2 matches, then Player 1 will take 2 matches,
— " — " — " —

if Player 2 takes 3 matches, then Player 1 will take 1 match,
— " — " — " —

- $n = 4j+3 = 4\cdot 1+3 = 7$, then Player 1 will take 2 matches, 5 will be left

similar plan as for the previous case $(n=6)$ follows.

- $n = 4j+1 = 4\cdot 1+1 = 5$ - Player 1 will lose:

if Player 1 takes 1 match, then Player 2 will take 3 matches, which will leave Player 1 with 1 match - Player 1 loses

if Player 1 takes 2 matches, Player 2 will take 2 matches, which will leave Player 1 again with 1 match - Player 1 loses

if Player 1 takes 3 matches Player 2 will take 1 match, which will leave Player 1 with 1 match. - Player 1 loses.

- we confirmed that $P(j)$ is true for $j=0$ and $j=1$.

<u>Inductive step:</u> assume that for any fixed $k \geq 1$ $P(i)$ is true for all i $0 \leq i \leq k$ $\Big\} (IH)$

consider $P(k+1)$:

- $n = 4(k+1) = 4k + 4$

$P(k)$ is true $^{(IH)}$, hence with $n = 4k$ Player 1 wins and at the last turn Player 2 was left with only 1 match.

So right now, there are $1+4=5$ matches in front of Player 2.

if Player 2 picks 1 match, then Player 1 picks 3 matches, and Player 2 is left with 1 match - he/she lost

if Player 2 picks 2 matches, then Player 1 picks 2 matches,
— " — " — " —

if Player 2 picks 3 matches, then Player 1 picks 1, and
— " — " — " —

• $n = 4(k+1) + 2 = 4k + 6 = (4k+2) + 4$

$P(k) \overset{(IH)}{\text{is}}$ true, hence this is the case when Player 1 wins, i.e. Player 2 was left with 1 match.

now Player 2 has $1+4=5$ matches to pick from:

- same as above, Player 2 will lose.

• $n = 4(k+1) + 3 = 4k+7 = (4k+3) + 4$

$P(k) \underset{(IH)}{\text{is}}$ true, hence this is the case when Player 1 wins, i.e.

– " – " – similar to the previous reasoning from now on

• $n = 4(k+1) + 1 = 4k+5 = (4k+1) + 4$

$P(k)$ is true by IH, so this is the case when Player 1 loses.

So right now there are $1+4=5$ matches in front of Player 1.

If Player 1 picks 1 match, then Player 2 will do 3, and Player 1 is left with 1 match – he/she lost

If Player 1 picks 2 matches, then Player 2 will pick 2, and
– " – " – " –

If Player 1 picks 3 matches, then Player 2 will pick 1, and
– " – " – " –

This is the case when Player 2 wins.

This completes the inductive step.

By ~~induction~~ strong induction we proved that $P(j)$ is true for all $j \in \mathbb{Z}$, $j \geq 0$.

q.e.d.