BRONX COMMUNITY COLLEGE
of the City University of New York

DEPARTMENT OF MATHEMATICS & COMPUTER SCIENCE

Review Sheet for CSI 35, Discrete Mathematics II.

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1. Prove by mathematical induction that for each positive integer n we have

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

2. Consider the binary relation $R = \{(x, y) \mid x = y + 1 \text{ or } x = y - 1\}$ on the set of all integers.

(a) Determine whether this relation is symmetric, reflexive, transitive, antisymmetric.

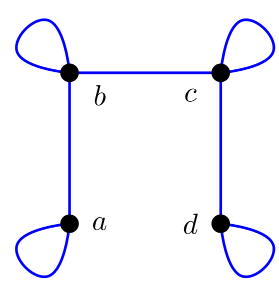
(b) Is it an equivalence relation? Is it a partial or total ordering?

3. Let $S = \{a, b, c, d\}$ and let R be the binary relation on S corresponding to the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Is R an equivalence relation? If it is, describe the equivalence classes and show how they partition S.

4. Let $S = \{a, b, c, d\}$ and let T be the binary relation on S corresponding to the graph



Is T an equivalence relation?

5. Can a simple graph exist with 13 vertices each of degree 3?

6. In the graph in Problem 4, find the number of vertices, the number of edges, the degree of each vertex. Identify whether it is a simple graph, a multigraph, or a pseudograph. Give an example of a proper subgraph. Identify any isolated and pendant vertices. Find its adjacency matrix and incidence matrix. Is it bipartite?

7. Draw the two simple graphs with the following adjacency matrices M and T:

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Are they bipartite? Are the two graphs isomorphic? If not explain why and if they are demonstrate the isomorphism.

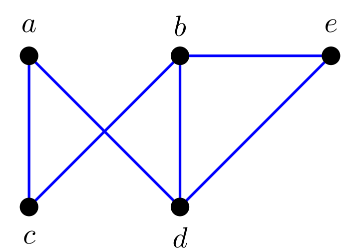
8. Each of the following lists of vertices form a path in the graph below. Which paths are simple? Which are circuits? What are the lengths of those that are paths? What are their end vertices?

(i) a, c, b, d, e

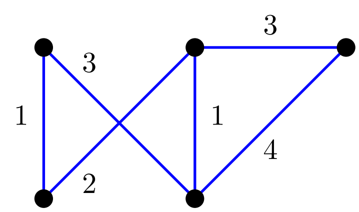
(ii) a, c, a, d, b, c, a

(iii) e, b, c, b, d, a

(iv) c, b, e, d, a, c



9. For the following weighted graph, find its minimum and maximum spanning trees.



10. For each of the graphs in Problems 4, 7, 8 determine whether it has an Euler path, Euler circuit, Hamiltonian path or a Hamiltonian circuit. Give examples if they exist.

11. For each of the graphs in Problems 4, 7, 8 determine whether it is planar. If so, draw it so that no edges cross. Find spanning trees for each of these graphs.

12. Give an example of a graph with chromatic number 100.

13. Use the *Handshaking Theorem* to prove or disprove the following: there exists a graph which has three vertices of degree 5, two vertices of degree 4, one vertex of degree 1, and a total of 11 edges.

14. Construct a binary search tree for the following words: eat, sip, hop, quack, ear, tap, fasten.

15. Construct a Huffman code for the following symbols and frequencies: A, 0.20; B, 0.25; C, 0.20; D, 0.15; E, 0.12; F, 0.08. What is the code for the string DABCF?

16. Evaluate the postfix expression: $2\ 4 * 3\ 2 \uparrow -\ 9\ *$

17. Give a recursive definition of each of the following sets:

(a) The set of rooted, full binary trees.

(b) The set of pairs of positive integers whose sum is odd.

(c) The set of bit strings.

18. For the following nine graphs: K_3, K_4, $K_{2,2}$, $K_{3,4}$, $K_{1,5}$, C_3, C_4, Q_2, W_3.

(a) Draw each of them.

(b) Say which are isomorphic.

(c) Give the chromatic number of each.

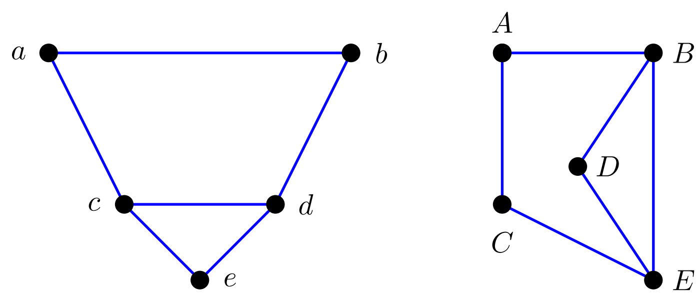
19. Let X be a set. Show that $(P(X), \subseteq)$ is a poset, where $P(X)$ denotes the power set of X and $\subseteq$ denotes subset.

20. Give a recursive definition for the sequence of squares: 1, 4, 9, 16, 25, ...

21. Define the function g recursively as follows: $g(2) = 5$ and $g(n) = -n + g(n-1)$ for all integers n. What is $g(4)$?

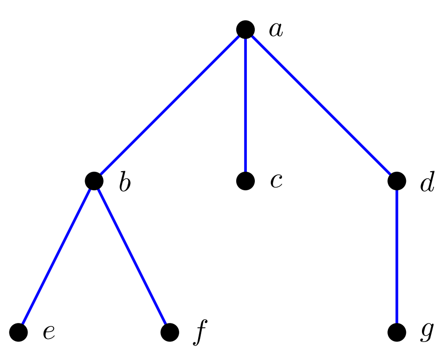
22. List several ways of showing that two graphs are not isomorphic.

23. Are these two graphs isomorphic? Prove the are by constructing an isomorphism f between them.



24. Construct the Hasse diagram for the poset $(\{2, 3, 4, 5, 8, 18\}, |)$, where $|$ means *divides*. Identify each of the following, or state that they do not exist: maximal elements, minimal elements, the greatest element, the least element.

25. Give the preorder, inorder and postorder traversals of the following tree:



26. Show in detail with an example how Dijkstra's algorithm works, finding the shortest path between to vertices in a weighted graph.

27. Calculate

(a) the number of internal vertices in a full 3-ary tree with 52 vertices;

(b) the maximum number of vertices at level three in a full binary tree.

28. The game of *Nim* is played by two players taking it in turns to remove stones from one of a number of piles. The last to play wins (this is the normal play convention).

(a) Draw the game tree for the Nim game starting with two piles where there are 3 stones in the first pile and just 1 stone in the second.

(b) Explain which of the first or second players has a winning strategy for the game in part (a).

29. For positive integers n, which is bigger 2^n or n^2? Make a conjecture. Prove your conjecture using induction.

30. Consider the binary relation $R = \{(1,1),(2,1),(2,3),(3,2)\}$ on the set $\{1,2,3,4\}$.

(a) Give its matrix and graph.

(b) Find R^2 and R^3.

(c) Determine whether this relation is symmetric, reflexive, transitive, antisymmetric.

Answers

1. Let $P(n)$ be the statement that the given equality is true for n. To prove $P(n)$ is true for all positive integers n we first check the basis step $P(1)$, i.e. that $\frac{1}{1\cdot 3} = \frac{1}{2\cdot 1+1}$ is true. Next we prove the induction step - that $P(k)$ implies $P(k+1)$. The left side of the equality in the statement of $P(k+1)$ equals

$$\frac{k}{2k+1} + \frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

using $P(k)$. This simplifies to $\frac{k+1}{2(k+1)+1}$ as required. Therefore, by induction, $P(n)$ is true for all positive integers n.

2. Of the four properties R is only symmetric. Therefore, it is not an equivalence relation and not a partial or total ordering.

3. This is an equivalence relation with two equivalence classes: $[a] = [b] = [c] = \{a,b,c\}$ and $[d] = \{d\}$.

4. T is reflexive and symmetric but not transitive. So it is not an equivalence relation.

5. No, the graph can not exist. The total degree must be even, so the number of odd degree vertices must be even.

6. The graph has 4 vertices, 7 edges. The vertex degrees are $3,4,4,3$. It is a pseudograph without any isolated (degree 0) or pendant (degree 1) vertices. An example of a proper subgraph is the graph containing the vertices c and d and just the loop on c. The adjacency and incidence matrices are,

$$\text{Adjacency: } \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \text{Incidence: } \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

where we ordered the edges clockwise. It is not bipartite (cannot be 2-colored).

7. Both are 2-colorable and hence bipartite. They are not isomorphic, having different numbers of edges.

8. (i) is a simple path of length 4, starting at a and ending at e, (ii) is a non-simple circuit of length 6, starting and ending at a, (iii) is a non-simple path of length 5, starting at e and ending at a, (iv) is a simple circuit of length 5, starting and ending at c.

9. The minimum weight spanning tree has weight 7. The maximum spanning tree weight is 12.

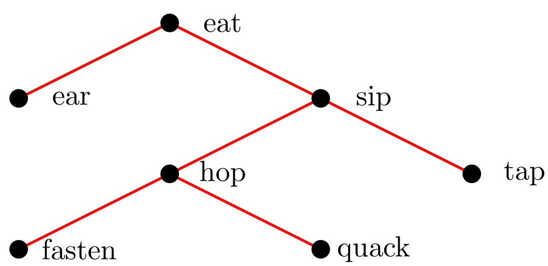
10. All four graphs have Euler and Hamilton paths. Only the graph corresponding to M in problem 7 has an Euler circuit. Only the graph corresponding to M in problem 7 and the graph in problem 8 have Hamilton circuits.

11. All four graphs are planar.

12. The complete graph K_{100} has chromatic number 100.

13. This graph does not exist: it has total degree $3 \cdot 5 + 2 \cdot 4 + 1 \cdot 1 = 24$ and, by the theorem, must have $24/2 = 12$ edges.

14.



15. A:000; B:01; C:10; D:001; E:110; F:111. The code for DABCF is 0010000110111.

16. -9

17. (a) Basis: the trivial tree is in the set. Recursion: if $T_1, \cdots, T_m$ are in the set, then so is the tree consisting of a root joined to subtrees $T_1, \cdots, T_m$. (b) Basis: $(0, 1)$ and $(1, 0)$ are in the set. Recursion: if (a, b) is in the set, so is $(a + 2, b)$ and $(a, b + 2)$. (c) Basis: the empty string is in the set. Recursion: if s is a string in the set, then so is $s0$ and $s1$.

18. (b) The graphs K_3 and C_3 are isomorphic, as are K_4 and W_3. The graphs $K_{2,2}$, C_4 and Q_2 are also isomorphic. (c) The chromatic numbers are 3, 4, 2, 2, 2, 3, 2, 2, 4 respectively.

19. The subset relation is reflexive, since any set is a subset of itself. The subset relation is anti-symmetric, since if A is a subset of B and B is a subset of A, then $A = B$. The subset relation is transitive since if A is a subset of B and B is a subset of C then A is a subset of C.

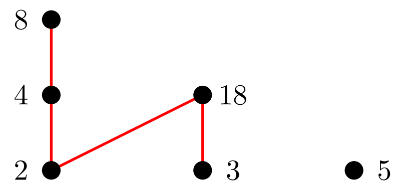
20. Basis: $a_1 = 1$. Recursion: $a_{n+1} = a_n + 2n + 1$.

21. -2

22. For example: different numbers of vertices; different numbers of vertices of a given degree; different numbers of edges; different numbers of circuits of a given length.

23. Set $f(a) = D$, $f(b) = A$, $f(c) = E$, $f(d) = B$, $f(e) = C$. We see that f is one-to-one and onto. Checking that the corresponding adjacency matrices agree proves that f is an isomorphism.

24. Maximal elements: 5, 8, 18; minimal elements: 2, 3, 5; the least and the greatest do not exist. The Hasse diagram is drawn below.



25. Preorder: a, b, e, f, c, d, g. Inorder: e, b, f, a, c, g, d. Postorder: e, f, b, c, g, d, a.

26. See the course text.

27. (a) 17; (b) 8.

28. Start with a vertex for the game position 3, 1. This will be the tree root. Label its children as 2,1 and 1,1 and 1 and 3. These are the first player's possible moves. Continue by drawing the next level of possible moves for the second player. The game ends when no stones are left and the leaves are labeled 0. We see that player 1 can force a win (move to 1,1).

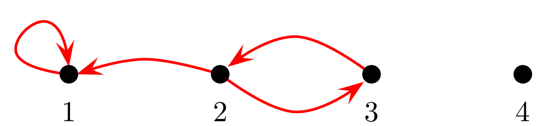
29. Let $P(n)$ be the statement: $2^n \geq n^2$. We prove $P(n)$ is true for all integers $n \geq 4$ as follows. The basis case $P(4)$ is true because $2^4 \geq 4^2$. Next we prove the induction step - that $P(k)$ implies $P(k+1)$. Multiplying both sides of the inequality of $P(k)$ by 2 shows that $2^{k+1} \geq 2k^2$. To prove $P(k+1)$ we need to show that $2^{k+1} \geq (k+1)^2$. So $P(k+1)$ follows if we can show that

$$2k^2 \geq (k+1)^2. \tag{1}$$

This is equivalent to $2k^2 \geq k^2 + 2k + 1$ and $k^2 - 1 \geq 2k$. We can write this as $(k-1)(k+1) \geq 2k$ and comparing factors it is clearly true for $k \geq 3$. (Note that we could also prove (1) with calculus.) Therefore, by induction, $P(n)$ is true for all integers $n \geq 4$.

30. (a) The relation's matrix and directed graph are

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



(b) We have $R^2 = \{(1,1), (2,1), (2,2), (3,1)(3,3)\}$ and $R^3 = \{(1,1), (2,1), (2,3), (3,1)(3,2)\}$.

(c) R has none of the four properties.

2004 AK, AW, 2010 AK. Revised 5/2011, **C.O'S.**