Chapter 10 Formulas / Theorems sheet

Type	Edges	Multiple edges allowed?	Loops allowed?
Simple graph	undirected	No	No
Multigraph	undirected	Yes	No
Pseudograph	undirected	Yes	Yes
Simple directed graph	directed	No	No
Directed multigraph	directed	Yes	Yes
Mixed graph	both types	Yes	Yes

[Theorem 1] The Handshaking Theorem
Let $G = (V,E)$ be an undirected graph with m edges.

Then $2m = \sum_{v \in V} deg(v)$

[Theorem 2] An undirected graph has an even number of vertices of odd degree.

[Theorem 3] Let G = (V,E) be a directed graph. Then $\sum_{v \in V} deg-(v) = \sum_{v \in V} deg+(v) = |E|$

[Def] a simple graph is called *bipartite* if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (i.e. no same-set vertices connections). We call the pair (V_1, V_2) a *bipartition* of V.

[Theorem 4] a simple graph is *bipartite* if and only if (iff) it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertics have the same color.

[Theorem] Hall's marriage theorem
The bipartite graph G = (V,E) with bipartition (V_1,V_2) has a *complete matching* from V_1 to V_2 if and only if (iff) $|N(A)| \geq |A|$ for all subsets A of V_1.

[Theorem] There is a simple path between every pair of distinct vertices of a connected undirected graph.

[Def] An *Euler circuit* in a graph G is a simple circuit containing every edge of G.
[Def] An *Euler path* in G is a simple path containing every edge of G

[Theorem] A connected multigraph with at least two vertices has an *Euler circuit* iff each of its vertices has even degree.

[Theorem] A connected multigraph has an *Euler path* but not an *Euler circuit* iff it has exactly two vertices of odd degree.

[Theorem] A directed graph has an Euler circuit iff all vertices with nonzero degree belong to a single strongly connected component and in-degree and out-degree of every vertex is same.

[Theorem] A directed graph has an Euler path iff
- at most one vertex has (out-degree) − (in-degree) = 1,
- at most one vertex has (in-degree) − (out-degree) = 1,
- every other vertex has equal in-degree and out-degree, and
- all of its vertices with nonzero degree belong to a single connected component of the underlying undirected graph.

[Def] A *Hamilton circuit* in a graph G is a simple circuit that passes through every vertex in G exactly once.
[Def] A *Hamilton path* in a graph G is a simple path that passes through every vertex in G exactly once.

Properties:
- A graph with a vertex of degree 1 cannot have a Hamilton circuit (each vertex is incident with two edges)
- If a vertex in the graph has degree 2, then both edges that are incident with this vertex must be part of the Hamilton circuit.
- When a Hamilton circuit is being constructed and it passed through a vertex, then all remaining edges incident with this vertex, other than the two used in the circuit, can be removed from consideration.
- A Hamilton circuit cannot contain a smaller circuit within it.

procedure Euler(G: multigraph...)
pick a vertex, say a
circuit := pick any simple circuit in G that starts at a
H := G with the edges of the *circuit* removed
while H has edges
  subcircuit := a circuit in H beginning at a vertex in H that
  also is an endpoint of an edge of the *circuit*.
  H := H with edges of *subcircuit* and all isolated vertices
  removed
  circuit := *circuit* with *subcircuit* inserted at the appropriate
  vertex
return *circuit* {*circuit* is an Euler circuit}

[Theorem] *Dirac's Theorem*
If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a *Hamilton circuit*.

[Theorem] *Ore's Theorem*
If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u)+\deg(v) \geq n$ for every pair of non-adjacent vertices u and v in G, then G has a *Hamilton circuit*.

procedure *Dijkstra*(G: weighted connected simple graph, with all weights positive)
{G=(V,E) has vertices $a = v_0, \ldots, v_n = z$, and weights $w(v_i,v_j)$, where $w(v_i,v_j) = \infty$ if $\{v_i,v_j\} \notin E$}

for i:= 1 **to** n   // set all the labels to infty
  $L(v_i) := \infty$
$L(a) := 0$, $S := \varnothing$ // set a's label to 0 and set S is empty infty

while $z \notin S$
  u := a vertex not in S with smallest L(u)
  $S := S \cup \{u\}$ // adding a vertex to S
  for all vertices v not in S   // update the labels; vertices are adjacent to u
    if L(u) + w(u,v) < L(v) **then** L(v) := L(u) + w(u,v)
return L(z) // return the length of a shortest path from a to z

[Def] a graph is called *planar* if it can be drawn in the plane without edges crossing.
Such a drawing is called a *planar representation of the graph*.

[Def] A *coloring of a simple graph* is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

[Def] A *chromatic number of a graph* is the least number of colors needed for coloring of this graph.
denotaion: $\chi(G)$

[Theorem] The Four Color Theorem
The chromatic number of a planar graph is no greater than four.