CSI 30 Fall 2021

HW 4 answers and solutions

Section 1.6

Exercise 1.6.1: Which expressions with predicates are propositions?

Predicates P, T and E are defined below. The domain of discourse is the set of all positive integers.

- P(x): x is even
- T(x, y): $2^x = y$
- E(x, y, z): $x^y = z$

Indicate whether each logical expression is a proposition. If the expression is a proposition, then give its truth value.

(e) E(6, 2, 36)

Proposition. E(6, 2, 36) is true because $6^2= 36$.

(f) E(2, y, 7)

Not a proposition because y is a variable.

(g) P(3) ∨ T(5, 32)

Proposition. P(3) ∨ T(5, 32) is true because T(5, 32) is true.

Exercise 1.6.2: Truth values for quantified statements about integers.

In this problem, the domain of discourse is the set of all integers. Which statements are true? If an existential statement is true, give an example. If a universal statement is false, give a counterexample.

(b) $\exists x\ (x + 2 = 1)$

True. Example: x = -1.

(d) $\forall x\ (x^2 - x \neq 0)$

False. Counterexample: x = 1 or x = 0.

(f) $\exists x\ (x^2 > 0)$

True. Example: any integer besides 0.

Exercise 1.6.3: Translating mathematical statements in English into logical expressions.

Consider the following statements in English. Write a logical expression with the same meaning. The domain of discourse is the set of all real numbers.

(c) There is a number that is equal to its square.

$\exists x\ (x^2 = x)$

(d) Every number is less than or equal to its square.

$\forall x\ (x \leq x^2)$

Exercise 1.6.4: Truth values for quantified statements for a given set of predicates.

The domain for this problem is a set {a, b, c, d}. The table below shows the value of three predicates for each of the elements in the domain. For example, Q(b) is false because the truth value in row b, column Q is F.

	P	Q	R
a	T	T	F
b	T	F	F
c	T	F	F
d	T	F	F

Which statements are true? Justify your answer.

(d) $\exists x\ Q(x)$

True. The element a is an example.

(e) $\forall x\ R(x)$

False. Elements a, b, c, and d are all counterexamples.

(f) $\exists x\ R(x)$

False. R(a), R(b), R(c), and R(d) are all false.

Section 1.7

Exercise 1.7.1: Translating quantified statements in English into logic.

In the following question, the domain of discourse is the set of employees at a company. One of the employees is named Sam. Define the following predicates:

- T(x): x is a member of the executive team
- B(x): x received a large bonus

Translate the following English statements into a logical expression with the same meaning.

(b) Everyone got a large bonus.

$\forall x\ B(x)$

(c) Sam did not get a large bonus even though he is a member of the executive team.

$\neg B(Sam) \land T(Sam)$

(d) Someone who is not on the executive team received a large bonus.

$\exists x\ (\neg T(x) \land B(x))$

(e) Every executive team member got a large bonus.

$\forall x\ (T(x) \rightarrow B(x))$

Exercise 1.7.2: Determining whether a quantified statement is a proposition.

Predicates P and Q are defined below. The domain of discourse is the set of all positive integers.

- P(x): x is prime
- Q(x): x is a perfect square (i.e., $x = y^2$, for some integer y)

Indicate whether each logical expression is a proposition. If the expression is a proposition, then give its truth value.

(b) $\forall x\ Q(x) \land \neg P(x)$

Not a proposition because the variable x in P(x) is not bound by the quantifier.

(d) $\exists x\ (Q(x) \land P(x))$

Proposition. The proposition is false because there is no positive integer that is a perfect square and prime.

Exercise 1.7.3: Translating quantified statements from English to logic, part 1.

In the following question, the domain of discourse is a set of students at a university. Define the following predicates:

- E(x): x is enrolled in the class
- T(x): x took the test

Translate the following English statements into a logical expression with the same meaning.

(b) All students enrolled in the class took the test.

$\forall x\ (E(x) \rightarrow T(x))$

(c) Everyone who took the test is enrolled in the class.
$\forall x\ (T(x) \rightarrow E(x))$

(d) At least one student who is enrolled in the class did not take the test.

$\exists x\ (E(x) \land \neg T(x))$

Exercise 1.7.4: Translating quantified statements from English to logic, part 3.

In the following question, the domain of discourse is a set of employees who work at a company. Ingrid is one of the employees at the company. Define the following predicates:

- S(x): x was sick yesterday
- W(x): x went to work yesterday
- V(x): x was on vacation yesterday

Translate the following English statements into a logical expression with the same meaning.

(g) Someone who missed work was neither sick nor on vacation.

$\exists x\ (\neg W(x) \land \neg S(x) \land \neg V(x))$

(h) Each person missed work only if they were sick or on vacation (or both).

$\forall x\ (\neg W(x) \rightarrow (S(x) \lor V(x)))$

(i) Ingrid was sick yesterday but she went to work anyway.

$S(\text{Ingrid}) \land W(\text{Ingrid})$

Exercise 1.7.5: Translating quantified statements from logic to English.

In the following question, the domain of discourse is the set of employees of a company. Define the following predicates:

- A(x): x is on the board of directors
- E(x): x earns more than $100,000
- W(x): x works more than 60 hours per week

Translate the following logical expressions into English:

(b) $\exists x\ (E(x) \land \neg W(x))$

There is someone who earns more than $100,000, but does not work more than 60 hours per week.

(f) $\exists x\ (A(x) \land \neg E(x) \land W(x))$

Someone on the board of directors does not earn more that $100,000 and works more than 60 hours per week.

Exercise 1.7.6: Determining whether a quantified logical statement is true.

In the following question, the domain of discourse is a set of male patients in a clinical study. Define the following predicates:

- P(x): x was given the placebo
- D(x): x was given the medication
- A(x): x had fainting spells
- M(x): x had migraines

Suppose that there are five patients who participated in the study. The table below shows the names of the patients and the truth value for each patient and each predicate:

	P(x)	D(x)	A(x)	M(x)
Frodo	T	F	F	T
Gandalf	F	T	F	F
Gimli	F	T	T	F
Aragorn	T	F	T	T
Bilbo	T	T	F	F

For each of the following quantified statements, indicate whether the statement is a proposition. If the statement is a proposition, give its truth value and translate the expression into English.

(b) $\exists x\ M(x) \land \exists x\ D(x)$

Proposition. True. One of the patients took the medication and one of the patients had migraines.

(e) $\forall x\ (M(x) \leftrightarrow A(x))$

Proposition. False. One of the patients had migraines, but no fainting spells.

(g) $\exists x\ (D(x) \land \neg A(x) \land \neg M(x))$

Proposition. True. One of the patients who took the medication had neither fainting spells nor migraines.

Section 1.8

Exercise 1.8.1: Applying De Morgan's law for quantified statements to logical expressions.

Apply De Morgan's law to each expression to obtain an equivalent expression in which each negation sign applies directly to a predicate. (i.e., $\exists x\ (\neg P(x) \lor \neg Q(x))$ is an acceptable final answer, but not $\neg\exists x\ P(x)$ or $\exists x\ \neg(P(x) \land Q(x))$).

(d) $\neg\forall x\ (P(x) \land (Q(x) \lor R(x)))$

$\exists x\ (\neg P(x) \lor (\neg Q(x) \land \neg R(x)))$

Exercise 1.8.2: Applying De Morgan's law for quantified statements to English statements.

In the following question, the domain of discourse is a set of male patients in a clinical study. Define the following predicates:

- P(x): x was given the placebo
- D(x): x was given the medication
- M(x): x had migraines

Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until each negation operation applies directly to a predicate and then translate the logical expression back into English.

Sample question: Some patient was given the placebo and the medication.

- $\exists x\ (P(x) \land D(x))$
- Negation: $\neg\exists x\ (P(x) \land D(x))$
- Applying De Morgan's law: $\forall x\ (\neg P(x) \lor \neg D(x))$
- English: Every patient was either not given the placebo or not given the medication (or both).

(b) Every patient was given the medication or the placebo or both.

- $\forall x\ (D(x) \lor P(x))$
- Negation: $\neg\forall x\ (D(x) \lor P(x))$
- Applying De Morgan's law: $\exists x\ (\neg D(x) \land \neg P(x))$
- English: There is a patient who was not given the medication and not given the placebo.

Exercise 1.8.3: Applying De Morgan's law for quantified statements to English statements.

In the following question, the domain of discourse is a set of students who show up for a test. Define the following predicates:

- $P(x)$: x showed up with a pencil
- $C(x)$: x showed up with a calculator

Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until each negation operation applies directly to a predicate and then translate the logical expression back into English.

Sample question: Every student showed up with a calculator.

- $\forall x\ C(x)$
- Negation: $\neg\forall x\ C(x)$
- Applying De Morgan's law: $\exists x\ \neg C(x)$
- English: Some student showed up without a calculator.

(c) Every student who showed up with a calculator also had a pencil.

- $\forall x\ (C(x) \rightarrow P(x))$
- Negation: $\neg\forall x\ (C(x) \rightarrow P(x))$
- Applying De Morgan's law: $\exists x\ (C(x) \land \neg P(x))$
- English: Some student showed up with a calculator and no pencil.
-

(d) There is a student who showed up with both a pencil and a calculator.

- $\exists x\ (P(x) \land C(x))$
- Negation: $\neg\exists x\ (P(x) \land C(x))$
- Applying De Morgan's law: $\forall x\ (\neg P(x) \lor \neg C(x))$
- English: Every student who showed up did not have a pencil or did not have a calculator (or both).

Exercise 1.8.4: Using De Morgan's law for quantified statements to prove logical equivalence.

Use De Morgan's law for quantified statements and the laws of propositional logic to show the following equivalences:

(a) $\neg\forall x\ (P(x) \land \neg Q(x)) \equiv \exists x\ (\neg P(x) \lor Q(x))$

$\neg\forall x\ (P(x) \land \neg Q(x))$	
$\exists x\ \neg(P(x) \land \neg Q(x))$	De Morgan's law
$\exists x\ (\neg P(x) \lor \neg\neg Q(x))$	De Morgan's law
$\exists x\ (\neg P(x) \lor Q(x))$	Double negation law

(c)
$\neg\exists x\ (\neg P(x) \lor (Q(x) \land \neg R(x))) \equiv \forall x\ (P(x) \land (\neg Q(x) \lor R(x)))$

$\neg\exists x\ (\neg P(x) \lor (Q(x) \land \neg R(x)))$	
$\forall x\ \neg(\neg P(x) \lor (Q(x) \land \neg R(x)))$	De Morgan's law
$\forall x\ (\neg\neg P(x) \land \neg(Q(x) \land \neg R(x)))$	De Morgan's law
$\forall x\ (P(x) \land \neg(Q(x) \land \neg R(x)))$	Double negation law
$\forall x\ (P(x) \land (\neg Q(x) \lor \neg\neg R(x)))$	De Morgan's law
$\forall x\ (P(x) \land (\neg Q(x) \lor R(x)))$	Double negation law