BRONX COMMUNITY COLLEGE
of The City University of New York

DEPARTMENT OF MATHEMATICS and COMPUTER SCIENCE

CSI 30 Discrete Mathematics I

Final Exam Review Questions. Answers and solutions.

Logic and Proofs. Answers.
Answers:

1. **a)** a proposition **b)** not a proposition (command)

2. **T** $\leftrightarrow$ **F** $\equiv$ **F**

3. **a)** $p \to q$ **b)** $q \wedge \neg p$ **c)** $p \wedge \neg q$

4. **a)** "The election is not decided, but the votes have been counted"
   b) "The election is decided if and only if the votes have been counted"
   c) "If the election is not decided then the votes haven't been counted"

5. **a)** "Yoshiko doesn't know Java or doesn't know calculus"
   b) "Rita won't move to Oregon and won't move to Washington"

6. The truth table for compound proposition $(\neg p \to \neg q) \wedge (q \leftrightarrow p)$:

p	q	$\neg p$	$\neg q$	$\neg p \to \neg q$	$q \leftrightarrow p$	$(\neg p \to \neg q) \wedge (q \leftrightarrow p)$
T	T	F	F	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	F	F
F	F	T	T	T	T	T

7. you can do it (you should see all Ts in the last column)!

8. $(\neg q \wedge (p \to q)) \to \neg p \equiv$
   $\equiv (\neg q \wedge (\neg p \vee q)) \to \neg p$ by law (11)
   $\equiv ((\neg q \wedge \neg p) \vee (\neg q \wedge q)) \to \neg p$ by Distributive law
   $\equiv ((\neg q \wedge \neg p) \vee F) \to \neg p$ by Negation law
   $\equiv (\neg q \wedge \neg p) \to \neg p$ by Identity law
   $\equiv \neg(\neg q \wedge \neg p) \vee \neg p$ by law (11)
   $\equiv (\neg\neg q \vee \neg\neg p) \vee \neg p$ by DeMorgan's law
   $\equiv q \vee p \vee \neg p$ by Double Negation law twice and omitting the parentheses
   $\equiv q \vee T$ by Negation law
   $\equiv T$ by Domination law

9. Let's start with a truth table first:

p	q	r	$p \to q$	$p \to r$	$(p \to q) \vee (p \to r)$	$p \to (q \vee r)$
T	T	T	T	T	T	T
T	T	F	T	F	T	T
T	F	T	F	T	T	T
T	F	F	F	F	F	F
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

Now let's use laws:

$(p \to q) \vee (p \to r) \equiv$
$\equiv (\neg p \vee q) \vee (\neg p \vee r)$ by law (11) twice
$\equiv \neg p \vee q \vee \neg p \vee r$ omitting parentheses
$\equiv \neg p \vee q \vee r$ by commutativity and law (5)
$\equiv p \to (q \vee r)$ by law (11) right to left

10. **a)** True, because we can provide such an n: if $n = 9$, then $n^2 = 81$

    b)False, when $n = 1$, $n^2 = n = 1$ - counterexample.

11. **a)** $\neg T(Flora)$ **b)** $\exists! x T(x)$

    c) $\neg \exists x C(x, Joe) \equiv \forall x \neg C(x, Joe)$

12. $\exists x \forall y (x \cdot y = 0)$

13. **a)** $\exists x \exists y\ \neg C(x, y)$

    b) $\exists x \forall y\ C(x, y)$

14. $\neg \exists y (\forall x\ \exists z\ T(x, y, z) \vee \exists x \forall z\ U(x, y, z)) \equiv \forall y (\exists x\ \forall z\ \neg T(x, y, z) \wedge \forall x \exists z\ \neg U(x, y, z))$

Sets

1. List the members of the following set.

   $\{x|x$ `is an integer such that` $x^2 < 25\}$

   Answer: $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

2. For sets $A = \{1, 3, 5, 7, 9\}$, $B = \{1, 2, 3, 5\}$, and $C = \{1, 3, 5, 7, 9, 11\}$, determine which of the following statements are true or false.

   a) $A \subseteq B$

   Answer (with explanation): False, because $7 \in A$, but $7 \notin B$

   b) $A \subseteq C$

   Answer (with explanation): True, because all elements of A are also elements of B.

   c) $B \subseteq C$

   Answer (with explanation): False, because $2 \in B$, but $2 \notin C$

3. Determine whether the following statements are true or false.

   a) $\underline{\mathbf{3}} \in \{\emptyset, \underline{\mathbf{3}}, \{3\}\}$

   Answer: True, see the highlighting with underlining.

   b) $\{3, 1\} \subseteq \{1, \{2\}, \{3\}, \{1, 3\}\}$ **Answer:** False, because 3 is not an element of set $\{1, \{2\}, \{3\}, \{1, 3\}\}$ (although $\underline{\mathbf{1}}$ is an element of set $\{\underline{\mathbf{1}}, \{2\}, \{3\}, \{1, 3\}\}$)

4. Let universe set $U = \{1, 2, 3, 4, 5, 6, 7\}$, and sets $A = \{1, 2\}$, $B = \{2, 3, 4\}$, $C = \{4, 5, 6, 7\}$. Find

   - $A \times C$

     Answer: $A \times C = \{(1, 4), (1, 5), (1, 6), (1, 7), (2, 4), (2, 5), (2, 6), (2, 7)\}$
   - $A \bigcap B \bigcap C$

     Answer: $A \bigcap B \bigcap C = \{2\} \bigcap \{4, 5, 6, 7\} = \emptyset$
   - $|C|$

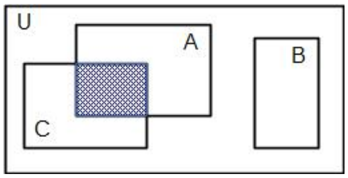
     Answer: we are asked to give *cardinality* of set C. $|C| = 4$
   - $\overline{B}$

     Answer: $\overline{B} = \{1, 5, 6, 7\}$
   - $A - B$

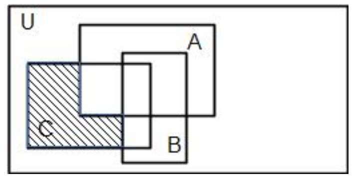
     Answer: $A - B = \{1\}$
   - $(A \bigcup C) \bigcap B$

     Answer: $(A \bigcup C) \bigcap B = \{1, 2, 4, 5, 6, 7\} \bigcap \{2, 3, 4\} = \{2, 4\}$

5. For the given sets (see the Venn Diagramm) show on the Diagramm



a) $A \bigcap C \bigcap \overline{B}$



b) $\overline{A \bigcup B} \bigcap C$

6. Use set identities to prove the following identity:

   $\overline{A \cup \overline{B}} = \overline{A} \cap B$

   Answer/proof: $\overline{A \cup \overline{B}}$ =by DeMorgan's law

   $= \overline{A} \cap \overline{\overline{B}} =$ by double negation law

   $= \overline{A} \cap B$

Functions

1. Determine whether the function $f(x) = x^5 + 1$, where $f(x) : Z \to Z$ is

   a) one-to-one?

   Answer: the function is *one-to-one* because no two different x-values are mapped to the same y-value.

   b) onto?

   Answer: The function is *not onto*, because $y = 3$ doesn't have a pre-image (same with $y = 4$ or $y = 5$.

   $f(1) = 1 + 1 = 2$, $f(2) = 2^5 + 1 = 32 + 1 = 33$,

   so values in between 2 and 33 (excluding) don't have a pre-image (domain of f is Z)

   c) bijection?

   Answer: The function $f(x)$ is not a bijection, because it is not onto (a function is bijective if it is both *one-to-one* and *onto*)

2. Determine the domain, the codomain/target, and the range of the function $f(x) = x^2+3$

   Answer:

   domain: R (set of all real numbers)

   codomain/target: R (set of all real numbers)

   range: $\{y \in R | \ y \geq 3\}$ (i.e. the set of all real numbers greater than or equal to 3)

3. Let $f(x) = 2x$ and $g(x) = 3x + 5$, are functions from **R** to **R**.

   a) Find $f \circ g$ and $g \circ f$

   Answer: $f \circ g = f(g(x)) = f(3x + 5) = 2(3x + 5) = 6x + 10$

   $g \circ f = g(f(x)) = g(2x) = 3(2x) + 5 = 6x + 5$

   b) Find $(fg)(x)$

   Answer: $(fg)(x) = (2x)(3x + 5) = 6x^2 + 10x$

   c) Find $(f + g)(x)$

   Answer: $(f + g)(x) = (2x) + (3x + 5) = 5x + 5$

4. Is function $g(x) = x^3 - 4$ invertible? If yes, explain why and find its inverse function. If no, provide explanations also.

   Solution:

   the function is *one-to-one* (i.e. no two different x-values are mapped to the same y-value), hence it is invertible. Let's find the inverse function $g^{-1}(x)$:

   $y = x^3 - 4$ - let's solve it for x: $x = \sqrt[3]{y + 4}$

   Therefore, $g^{-1}(x) = \sqrt[3]{x + 4}$

Algorithms

1. Give an algorithm, using pseudocode, that takes a list of n distinct integers as input and finds the location of the largest odd integer in the list. If there is no such integer, the algorithm should return 0.

   Answer:

   Input: $a_1, \ldots, a_n$: integers

   Output: location of the largest odd integer or 0

```
procedure largest_odd(a_1,...,a_n)
location := 0
largestOdd := a_1
For i:= 1 to n
    if (a_i is odd and a_i >= largestOdd), location := i
End-for
Return(location)
```

2. Present the algorithm, using a pseudocode, of finding the largest integer in an unordered sequence of n integers.

   Don't forget to describe the input and output for your algorithm.

   Answer:

   Input: $a_1, \ldots, a_n$: integers

   Output: largest integer

```
procedure largest(a_1,...,a_n)
largest := a_1
For i:= 1 to n
    if (a_i > largest), largest := a_i
End-for
Return(largest)
```

3. Take a look at the following algorithm:

```
procedure it(n:positive integer)
sum := 0
For i:= 1 to n
    sum := sum + i*i
End-for
Return(sum)
```

   (a) How many multiplication operations will be done (an expression with `n`)?

   Answer: n

   (b) If $n = 3$, what value will be returned?

   Answer: 14 (because $1^2 + 2^2 + 3^2 = 14$)

4. Given the algorithm:

```
procedure thing(a_1,a_2,a_3,...a_n:integers)
sum1 := 0
sum2 := 0
For i := 1 to n
   If (a_i > 0), sum1 := sum1 + a_i
   If (a_i < 0), sum2 := sum2 + a_i
End-for
Return(sum1,sum2)
```

For the set of values {1,5,-2,-9,2,5,-7} as input for the above algorithm, what are values of `sum1` and `sum2` that will be returned?

Answer: $sum1 = 1 + 5 + 2 + 5 = 13$, and $sum2 = (-2) + (-9) + (-7) = -18$

5. Consider the algorithm:

```
procedure foo(n:integer)
If n > 10, print('A')
If (n <= 10) and (n > -10), print('B')
Else print('C')
```

a) What will be printed if the procedure `foo` is run on `n=4`?

Answer: B

b) What will be printed if the procedure `foo` is run on `n=-4`?

Answer: B

c) What will be printed if the procedure `foo` is run on `n=24`?

Answer: A

6. Use Linear search to find 13 in the following list: $1, 7, 2, 3, 6, 8, 13, 4, 89$

How many comparisons will be performed?

Answer: 7 comparisons

7. Can binary search be used if it gets the following list as input: $1, 7, 2, 3, 6, 8, 13, 4, 89$?

Answer: No, the input elements must be in increasing order

8. Use binary search to find 14 in the following list: $1, 6, 8, 9, 13, 14, 16, 22, 36, 38$

Show all the splits and all the middle elements.

Solution:

$1, 6, 8, 9, 13, 14, 16, 22, 36, 38,$

middle element $= 13$ (10 `div` $2 = 5$, i.e. 5^{th} element):

$14! = 13$, $14 > 13$ hence consider upper half (excluding the middle element)

$14, 16, 22, 36, 38,$

middle = 22:

$14! = 22$, $14 < 21$, hence consider the lower half (excluding the middle element)

14, 16,

middle = 14:

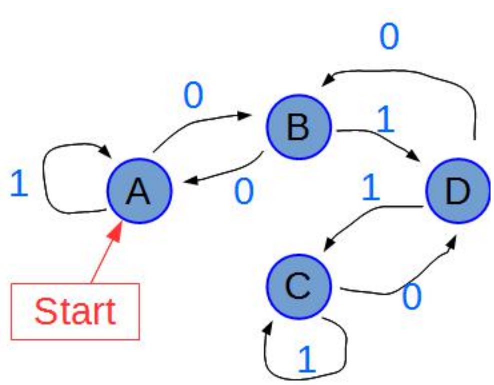
$14 = 14$. We found the value!

the algorithm will return the position of the element.

9. If binary search is used to find 10 in the following list: 9, 10, 14, 16, 22, 36, 56, 59, 61
   How many splits will be performed before the element is found?
   Answer: 1 split
   splits:
   9, 10, 14, 16, 22, 36, 56, 59, 61 (starting with)
   9, 10, 14, 16 (after the first split)
   then middle element is 10 - we found it

10. Use **bubble sort** algorithm to sort the list 5, 2, 4, 1, 3
    (show all the passes, with interchanges, see our lecture slides)
    How many interchanges will be performed during the first pass?
    Answer: there will be n-1 = 4 passes. In the first pass there will be 4 interchanges.

11. For the following FSM



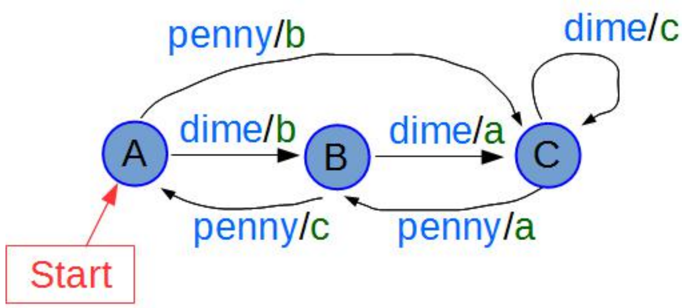
1) What is the current state after the FSM has processed the input sequence `0 1 0 1 1`?

Answer: C

2) What input sequence required to get from state A to state A and changing to at least one other state?

Answer: `0 0` or `0 1 0 0` and so forth

12. For the following FSM



1) What is the current state after the FSM has processed the input sequence `PENNY DIME PENNY DIME`?

Answer: C

2) What input sequence required to get from state A to state A and changing to at least one other state?

Answer: `penny penny penny`

Integers

1. Find

   a) 243 **div** 13 **b)** 243 **mod** 13 **c)** (-100) **div** 23 **d)** (-100) **mod** 23

   Answer:

   a) 18 **b)** 9 $(243 - 13 \times 18 = 9)$

   c) -5 $(-100 \div 23 = -4.347\ldots$, take one less) **d)** 15 $((-100) - 23 \times (-5) = 15)$

2. State True or False, explain why.

   $12 \equiv 54(\textbf{mod } 7)$

   Solution:

   we will use a theorem that says that $a \equiv b(\texttt{ mod } n)$ **iff** (is and only if) $a \texttt{ mod } n = b \texttt{ mod } n$

   12 **mod** $7 = 5$

   54 **mod** $7 = 5$

   Hence, $12 \equiv 54(\textbf{mod } 7)$

3. Give prime factorization of 231, if possible.

   Solution: $231 \div 3 = 77$, hence $231 = 3 \times 6 \times 11$

   Answer: $231 = 3 \cdot 6 \cdot 11$

4. Check if 421, 387, and 157 are prime numbers

   (use Theorem 6.4.1: Small factors from zyBooks)

   Solution:

   1) to check if 421 is prime or not we need to check all prime numbers <= $\sqrt{421} \approx 20$: $2, 3, 5, 7, 11, 13, 17$, and 19. None of them divide 421, hence the number 421 is a prime number.

   2) to check if 387 is prime or not we need to check all prime numbers <= $\sqrt{387} \approx 19$: $2, 3, 5, 7, 11, 13$, and 17.

   Note that the sum of digits of 387 is 18, which is divisible by 3. Hence 387 is divisible by 3!

   $387 \div 3 = 129$, hence the number 387 is not a prime number.

   3) to check if 157 is prime or not we need to check all prime numbers <= $\sqrt{157} \approx 12$: $2, 3, 5, 7$, and 11. None of them divide 157, hence the number 157 is a prime number.

   Answer: 421 and 157 are prime numbers, and 387 is not.

5. Find $gcd(231, 42)$ and $lcm(231, 42)$

   Solution:

   $231 = 3 \times 7 \times 11$ $42 = 2 \times 3 \times 7$,

   hence $gcd(231, 42) = 3 \times 7 = 21$, and

$lcm(231, 42) = 2 \times 3 \times 7 \times 11 = 462$

Answer: $gcd(231, 42) = 21$, $lcm(231, 42) = 462$

6. Check if the given pairs are relatively prime

   a) 42, 154 **b)** 396, 175

   Solution:

   a) $gcd(42, 154)$ is at least 2 (they are both divisible by 2), hence they are not relatively prime

   b) $gcd(396, 175) = 1$, hence they are relatively prime

   computations:

   $396 = 2^2 \times 3^2 \times 11$ and $175 = 5^2 \times 7$

   they don't have common factors! $gcd(396, 175) = 1$

7. Find the expansion base 5 of 187.

   Solution:

   $187 \div 5 = 37$ R 2

   $37 \div 5 = 7$ R 2

   $7 \div 5 = 1$ R 2

   STOP.

   List the last quotient (1) and all the remainders (in reverse order): $187 = (1222)_5$

   Answer: $187 = (1222)_5$

8. Find the expansion base 16 of 187.

   Solution:

   $187 \div 16 = 11$ R 11

   $11 \div 16 = 0$ R 11

   STOP

   List the last quotient (0) and all the remainders (use letter for anything > 9) (in reverse order):

   $187 = (0\mathtt{BB})_{16} = (\mathtt{BB})_{16}$

   Answer: $187 = (\mathtt{BB})_{16}$

9. Find the binary of 49.

   Solution:

   $49 \div 2 = 24$ R 1

   $24 \div 2 = 12$ R 0

   $12 \div 2 = 6$ R 0

   $6 \div 2 = 3$ R 0

   $3 \div 2 = 1$ R 1

STOP.

List the last quotient (1) and all the remainders (in reverse order): $49 = (110001)_2 = (0011\ 0001)_2$

Answer: $49 = (11\ 0001)_2$

10. Compute the decimal representation of $(012112)_3$

    Solution:

    $(012112)_3 = 2 \times 3^0 + 1 \times 3^1 + 1 \times 3^2 + 2 \times 3^3 + 1 \times 3^4 = 149$

    Answer: $(012112)_3 = 149$

11. Compute the decimal representation of $(77501)_8$

    Solution:

    $(77501)_8 = 1 \times 8^0 + 0 \times 8^1 + 5 \times 8^2 + 7 \times 8^3 + 7 \times 8^4 = 32577$

    Answer: $(77501)_8 = 32577$

12. Compute the decimal representation of $(A501)_{12}$

    digit A represents 10, and digit B represents 11.

    Solution:

    $(A501)_{12} = 1 \times 12^0 + 0 \times 12^1 + 5 \times 12^2 + 10 \times 12^3 = 18001$

    Answer: $(A501)_{12} = 18001$

13. Compute 5^6 **mod** 7 without use of calculator

    Solution: 5^6 **mod** $7 =$

    $= [((5 \times 5) \texttt{ mod } 7) \times ((5 \times 5) \texttt{ mod } 7) \times ((5 \times 5) \texttt{ mod } 7)] \texttt{ mod } 7 =$

    $= [4 \times 4 \times 4] \texttt{ mod } 7 =$

    $= [(4 \times 4) \texttt{ mod } 7 \times 4] \texttt{ mod } 7 =$

    $= [2 \times 4] \texttt{ mod } 7 = 1$

    Answer: 5^6 **mod** $7 = 1$

14. What is the value of $26 + 28$ in Z_{30}?

    Answer: $26 + 28$ in $Z_{30} = 54 \texttt{ mod } 30 = 24$

15. What is the value of 16×20 in Z_{30}?

    Answer: 16×20 in $Z_{30} = 320 \texttt{ mod } 30 = 20$

16. Check which integers are multiplicative inverses of $5 \texttt{ mod } 59$.

    a) 12

    Solution: $(12 \times 5) \texttt{ mod } 59 = 60 \texttt{ mod } 59 = 1$

Hence 12 is a multiplicative inverse of $5 \bmod 59$.

b) 46

Solution: $(46 \times 5) \bmod 59 = 230 \bmod 59 = 53$

Hence 46 is not a multiplicative inverse of $5 \bmod 59$.

c) 71

Solution: $(71 \times 5) \bmod 59 = 355 \bmod 59 = 1$

Hence 71 is a multiplicative inverse of $5 \bmod 59$.

17. Compute $(351^8 + 86)$ **mod** 10 without use of calculator

    Solution: $(351^8 + 86)$ **mod** $10 =$

    $((351 \bmod 10)^8 + 86 \bmod 10)$ **mod** $10 =$

    $(1^8 + 6) \bmod 10 = 7 \bmod 10 = 7$

    Answer: $(351^8 + 86)$ **mod** $10 = 7$

18. Compute $[(43 \bmod 10) + (76 \bmod 10)] \bmod 10$

    Solution:

    $[(43 \bmod 10) + (76 \bmod 10)] \bmod 10 =$

    $[(3) + (6)] \bmod 10 = 9 \bmod 10 = 9$

    Answer: $[(43 \bmod 10) + (76 \bmod 10)] \bmod 10 = 9$

19. Consider the following ***linear congruential generator*** of pseudo-random numbers:

    $x_n = (5x_{n-1} + 7) \bmod 12$, with seed $x_0 = 4$

    What sequence of pseudo-random numbers does it generate?

    Solution:

    $x_1 = (5 \times 4 + 7) \bmod 12 = 27 \bmod 12 = 3$

    $x_2 = (5 \times 3 + 7) \bmod 12 = 22 \bmod 12 = 10$

    $x_3 = (5 \times 10 + 7) \bmod 12 = 57 \bmod 12 = 9$

    $x_4 = (5 \times 9 + 7) \bmod 12 = 52 \bmod 12 = 4$ STOP. this was our seed x_0

    Pattern: $4, 3, 10, 9$

    Here is the sequence of pseudo-random numbers that will be generated: $4, 3, 10, 9, 4, 3, 10, 9, 4, 3, \ldots$

    Answer: $4, 3, 10, 9, 4, 3, 10, 9, 4, 3, \ldots$

20. Consider a simple ***cryptosystem*** in which the set of all possible *plaintexts* m come from Z_N for some integer N. Alice and Bob share a secret number $k \in Z_N$. The security of their encryption scheme rests on the assumption that no one besides them knows the number k. To encrypt a *plaintext* $m \in Z_N$, Alice computes:

    $c = (m + k) \bmod N$ (encryption)

    Alice sends the *ciphertext* c to Bob. When Bob receives the *cyphertext* c, he decrypts c as follows:

$m = (c - k) \texttt{ mod } N$ (decryption)

Assume that $N = 650$, and Alice and Bob agreed on key $k = 127$

a) Alice wants to send the *plaintext* message $m = 345$ to Bob.

What will be the encrypted message, *cyphertext* c?

Solution:

$c = (m + k) \texttt{ mod } N = (345 + 127) \texttt{ mod } 650 = 472 \texttt{ mod } 650 = 472$

Answer: $c = 472$

b) Bob received *cyphertext* $c = 534$ from Alice.

What will be the *plaintext* message m after he decrypts the *cyphertext*?

Solution:

$m = (c - k) \texttt{ mod } N = (534 - 127) \texttt{ mod } 650 = 407 \texttt{ mod } 650 = 407$

Answer: $m = 407$

c) Suppose that Eve somehow found out that $N = 650$ and also managed to learn that message $m = 555$ corresponds to $c = 32$. Can she infer/deduce the value for key k? Show how.

Solution:

Eve knows $N = 650$, $m = 555$ and $c = 32$. She also knows that Alice and Bob are using a simple ***cryptosystem***. In particular, she knows the encryption formula: $c = (m + k) \texttt{ mod } N$

She will substitute everything she knows into it:

$32 = (555 + k) \texttt{ mod } 650$,

in other words $(555 + k) - 650 = 32$, from which she will find that $k = 127$

Counting

1. At a bakery, there are three shapes of bread that can be made: rolls, bowl, loaf, and extra large loaf. Each bread can be of type: Ciabatta, Country, Whole Grain, Classic White, and Focaccia. Each bread can be sliced or not. How many different choices of bread are available?

   Solution: we will use *product rule*

   4 shapes, 5 types, 2 conditions (sliced or not)

   $4 \times 5 \times 2 = 40$

   Answer: 40 choices of bread are available

2. How many bit-strings of length 16 are there, that start with 0 or end with 11?

   Solution: "special " bits are fixed, so we will use product rule for each case (start with 0, end with 11) here as well

   Start with 0:

   $\underline{0}$ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

   $1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^{15} = 32768$

   End with 11:

   _ _ _ _ _ _ _ _ _ _ _ _ _ _ $\underline{1}$ $\underline{1}$

   $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 1 \times 1 = 2^{14} = 16384$

   What did we count twice:

   $\underline{0}$ _ _ _ _ _ _ _ _ _ _ _ _ _ $\underline{1}$ $\underline{1}$

   $1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 1 \times 1 = 2^{13} = 8192$

   We will use Exclusion-Inclusion principle to count bit-strings of length 16 that start with 0 or end with 11:

   $32768 + 16384 - 8192 = 40960$

   Answer: $40,960$ bit-strings of length 16 that start with 0 or end with 11?

3. List all 3-permutations and 3-combinations (3-subsets) of the set $S = \{1, 2, 3, 4, 5\}$.

   Solution:

   1) for permutations: there should be $P(5,3) = \dfrac{5!}{(5-3)!} = \dfrac{5!}{(2)!} = 60$ of them

   $\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,1,3\}, \{2,1,4\}, \{2,1,5\}, \{2,3,1\}, \{2,3,4\}, \{2,3,5\}, \{2,4,1\}, \{2,4$

   too many to list!

   2) for combinations: there should be $C(5,3) = \binom{5}{3} = \dfrac{5!}{3!(5-3)!} = 10$ of them

   {1,2,3 },{1,2,4 },{1,2,5 },{1,3,4 },{1,3,5 },{1,4,5 },{2,3,4 },{2,3,5 },{2,4,5 },{3,4,5 },

4. Find the values of $P(16,5)$ and $C(26,4) = \binom{26}{4}$.

   Solution:

$P(16,5) = \dfrac{16!}{(16-5)!} = \dfrac{16!}{11!} = 16 \times 15 \times 14 \times 13 \times 12 = 524,160$

$C(26,4) = \dbinom{26}{4} = \dfrac{26!}{(26-4)!4!} = \dfrac{26!}{22!4!} = \dfrac{26 \times 25 \times 24 \times 23}{4 \times 3 \times 2} = 14,950$

Answer: $P(16,5)524,160$ and $C(26,4) = \binom{26}{4} = 14,950$

5. How many bit-strings of length 14 contain

   (a) five 0's in the beginning?

   Solution:

   $\underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0}$ _ _ _ _ _ _ _ _ _

   there are $2^9 = 512$ bit strings of length 14 that contain five 0's in the beginning

   (b) at least four 1's?

   Solution: The order of digits is important, but we do not distinguish between 1's, only positions of those 1's are of importance to us. Hence we can re-phrase that n places in the string (out of 14) are occupied with 1's. Therefore, we will use combinations.

   at least four means 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 and 14

   -way too many to count. Let's follow the compliment principle.

   The negation of " at least four 1's " is " at most 3 1's ", therefore we need to find $C(14,0) + C(14,1) + C(14,2) + C(14,3) = \binom{14}{0} + \binom{14}{1} + \binom{14}{2} + \binom{14}{3} =$ $\dfrac{14!}{0!(14-0)!} + \dfrac{14!}{1!(14-1)!} + \dfrac{14!}{2!(14-2)!} + \dfrac{14!}{3!(14-3)!} = \dfrac{14!}{14!} + \dfrac{14!}{13!} + \dfrac{14!}{2!12!} + \dfrac{14!}{3!11!} =$ $1 + 14 + 91 + 364 = 470$

   There are 470 strings of length 14 that contain at most 3 1's.

   There are 2^{14} strings of length 14.

   Hence there are $2^{14} - 470 = 15,914$ bit strings of length 14 that contain at least four 1's

   Answer: $15,914$ bit strings of length 14 that contain at least four 1's

   (c) number of 1's and the number of 0's are the same?

   Solution:

   it means that we have 7 1s and 7 0's.

   $C(14,7) = \dbinom{14}{7} = \dfrac{14!}{7!(14-7)!} = \dfrac{14!}{7!7!} = 3,432$

   Answer: There are $3,432$ bit strings of length 14 that have equal number of 1's and 0's.

6. In how many ways can the letters of the word 'COLUMN' be arranged?

   Solution: this is from Section 7.8

   We have 6 places, and there are no letter repetitions. Hence:

   6 places to put one C

   5 places to put one O

   4 places to put one L

3 places to put one U

2 places to put one M

1 place to put one N

$$\binom{6}{1} \times \binom{5}{1} \times \binom{4}{1} \times \binom{3}{1} \times \binom{2}{1} \times \binom{1}{1} = \frac{6!}{1!5!} \times \frac{5!}{1!4!} \times \frac{4!}{1!3!} \times \frac{3!}{1!2!} \frac{2!}{1!1!} \times \frac{1!}{1!0!} = 6! = 720$$

or

first place: 6 options

second place: 5 options

third place: 4 options

fourth place: 3 options

fifth place: 2 options

sixth place: 1 option

$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

Answer: 720 ways to arrange the letters of the word 'COLUMN'

7. In how many ways can the letters of the word 'ABRAKADABRA' be arranged?

   Solution:

   11 places to put five A's

   6 places to put two B's

   4 places to put two R's

   2 places to put one K

   1 places to put one D

   $$\binom{11}{5} \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{1} \times \binom{1}{1} = \frac{11!}{5!6!} \times \frac{6!}{2!4!} \times \frac{4!}{2!2!} \times \frac{2!}{1!1!} \times \frac{1!}{1!0!} = \frac{11!}{5!2!2!} = 83,160$$

8. There are 10 men and 8 women in a room. In how many ways we can choose 4 men and 4 women from the room?

9. How many integers from 1 to 50 are only multiples of 2 or 3?

10. In a group of 50 students 24 like cold drinks and 36 like hot drinks and each student likes at least one of the two drinks. How many like both coffee and tea?

11. How many passwords of length 5, that contain only lower case letters and four special symbols $`, -, +, \_\_$ can be made?