Chapter 6. Integer properties

6.1 The Division Algorithm

6.2 Modular arithmetic

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a is called a factor of b,
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denotation: $a \mid b$ "a divides b"
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if we use Predicate Logic: $\exists c\,(a \cdot c = b)$, with domain: all integers

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Determine whether $7 \mid 24$, and $7 \mid 168$

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$7 \nmid 24$ *because* $24 \div 7 \approx 3.43$ – *not an integer*

$7 \mid 168$ *because* $168 \div 7 = 24$, $c = 24$
– there is an integer such that $7 \cdot 24 = 168$

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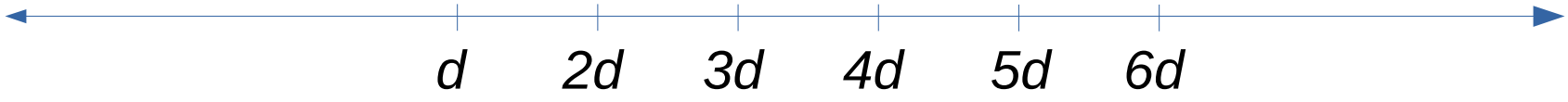

$dk \leq n$

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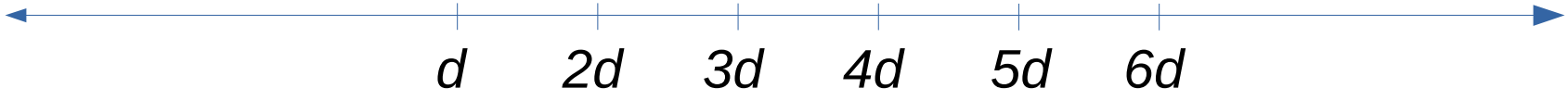

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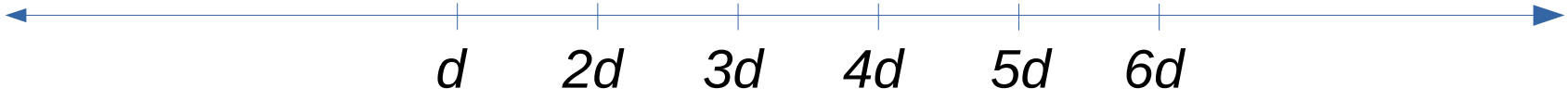

$dk \le n$ or $k \le \frac{n}{d}$ and finally $k = \left\lfloor \frac{n}{d} \right\rfloor$ ← that many

[Theorem 1] *Basic properties of divisibility*
Let a, b, and c be integers. Then
(1) if $a \mid b$ and $a \mid c$, then $a \mid (b+c)$
(2) if $a \mid b$, then $a \mid bc$, for all integers c
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Examples:
(1) $2 \mid 4$, $2 \mid 8$, then $2 \mid (4+8)$ i.e. $2 \mid 12$
(2) $2 \mid 4$, then $2 \mid 4*2$, $2 \mid 4*3$, $2 \mid 4*4$, …
(3) $2 \mid 4$, $4 \mid 8$, then $2 \mid 8$

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(1) If $a \mid b$ then there exists an integer s such that $b = a \cdot s$,
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Then $b + c = as + ak = a(s+k)$

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(2) left as homework suggested practice

qed

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[Corollary]
If a, b, and c are integers such that $a \mid b$ and $a \mid c$, then $a \mid (mb + nc)$ for any integers m and n.

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[Corollary]
If a, b, and c are integers such that $a \mid b$ and $a \mid c$, then $a \mid (mb + nc)$ for any integers m and n.

Example: $3 \mid 6$ and $3 \mid 9$, then $3 \mid (2 \cdot 6 + 4 \cdot 9)$ or $3 \mid 48$

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[Theorem 1] *Basic properties of divisibility*
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So, $c = b \cdot k = a \cdot s \cdot k$, Therefore $a \mid c$

(2) left as homework assignment

qed

[Corollary]
If a, b, and c are integers such that $a \mid b$ and $a \mid c$, then $a \mid (mb + nc)$ for any integers m and n.

Proof: if $a \mid b$, then by (2) from Theorem 1, $a \mid mb$

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So, $c = b \cdot k = a \cdot s \cdot k$, Therefore $a \mid c$

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[Corollary]
If a, b, and c are integers such that $a \mid b$ and $a \mid c$, then $a \mid (mb + nc)$ for any integers m and n.

Proof: if $a \mid b$, then by (2) from Theorem 1, $a \mid mb$
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[Corollary]
If a, b, and c are integers such that $a \mid b$ and $a \mid c$, then $a \mid (mb + nc)$ for any integers m and n.

Proof: if $a \mid b$, then by (2) from Theorem 1, $a \mid mb$
if $a \mid c$, then by (2) from Theorem 1, $a \mid nc$
Therefore, $a \mid (mb + nc)$, by (1) from Theorem 1

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Let a be an integer, and d be a positive integers. Then there are unique integers q and r, with $0 \leq r < d$, such that $a = dq + r$

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$q = a$ **div** d
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dividend *divisor* *quotient* *remainder*

```
  q R r
d ) a
  .....
```

or

```
a | d
..| q R r
```

notation:
$q = a$ **div** d
$r = a$ **mod** d

```
   19
7 ) 134
  - 7
    64
  - 63
     1
```

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Example 3:
$134 \div 7 = 19$ **R** 1

a (dividend) *d (divisor)* *q (quotient)* *r (remainder)*

```
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Example 3: $a = d \cdot q + r$

$134 \div 7 = 19$ **R** 1 or $134 = 7 \cdot 19 + 1$

a (dividend) *d (divisor)* *q (quotient)* *r (remainder)*

```
   19
7 ) 134
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     1
```

As you probably noticed, Theorem 2 is not actually an algorithm, but it is its traditional name, so we'll use it.

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Let's check: $113 = 11 \cdot 10 + 3 = 110 + 3 = 113$

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What are the quotient and remainder when *-17* is divided by *4*?

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What are the quotient and remainder when *-17* is divided by *4*?

Solution: -17 ÷ 4
divisor *d* is a positive integer, therefore quotient *q* will carry the sign, also remainder *r* is a positive integer, $0 \leq r < d$

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r = -17 **mod** $4 = 1$ or -1 ?

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$r = -17$ **mod** $4 = 1$ or -1 ? – remainder $r = -17 - (-20) = 3$

Let's check: $-17 = 4 \cdot (-4) + 1 = -16 + 1 = -15$

Example 4:
What are the quotient and remainder when *113* is divided by *11*?

Solution:
$q = 113$ **div** $11 = 10$ – quotient
$r = 113$ **mod** $11 = 3$ – remainder

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Example 5:
What are the quotient and remainder when *-17* is divided by *4*?

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Answer: q = -5, r = 3

Recall $q = a$ ***div*** d, q is a quotient
$r = a$ ***mod*** d, r is a remainder

6.2 Modular arithmetic

Recall $q = a \textbf{ div } d$, q is a quotient
$r = a \textbf{ mod } d$, r is a remainder

Karl Friedrich Gauss developed the concept of congruence (in 18th century). The notion of congruence played an important role in development of number theory

[Def. 1]
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Example 1:
Let $a = 25$, $b = 19$, and $m = 3$.

6.2 Modular arithmetic

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$r = a$ ***mod*** d, r is a remainder

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[Def. 1]
Let *a,b* be integers, and *m* be a positive integer. *a* is congruent to *b* modulo *m*, if *m* | *(a-b)*

Notation: $a \equiv b$ (**mod** *m*) " *a* is congruent to *b* modulo *m* "
$a \not\equiv b$ (**mod** *m*) " *a* is not congruent to *b* modulo *m* "

- we use the notation of congruency to indicate that two numbers have the same remainders when divided by *m*.

Example 1:
Let $a = 25$, $b = 19$, and $m = 3$.

25 is congruent to *19* modulo *3*, because *3* | *(25-19)*, i.e. *3* | *6*
Note that *25* ***mod*** *3* = *1*, and *19* ***mod*** *3* = *1*

[Theorem 3]
Let *a,b* be integers, and *m* be a positive integer. Then $a \equiv b \pmod{m}$ iff (if and only if) $a \bmod m = b \bmod m$.

Recall the example 1's note: $25 \bmod 3 = 1$, and $19 \bmod 3 = 1$

[Theorem 3]
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Proof:
1) Let's prove $\Rightarrow$: proof will be here

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Proof:
1) Let's prove $\Rightarrow$: proof will be here

2) Let's prove $\Leftarrow$: assume that $a \bmod m = b \bmod m = r$, $r > 0$,
then $\exists$q,s – intgers ($a = mq + r$ and $b = ms + r$).

From there, $a-b = mq + r - (ms + r) = mq + r - ms - r = mq - ms = m(q-s)$
Therefore m | (a-b).

qed

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Let *a,b* be integers, and *m* be a positive integer. Then $a \equiv b \pmod{m}$ iff (if and only if) $a \bmod m = b \bmod m$.

Example 2:
Determine whether *17* is congruent to *24* modulo *7* and whether *24* and *14* are congruent modulo *6*.

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The questions posed: $17 \equiv 24 \pmod{7}$?
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1) $17 \bmod 7 = 3$, and $24 \bmod 7 = 3$, therefore $17 \equiv 24 \pmod{7}$.

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Let a,b be integers, and m be a positive integer. Then $a \equiv b \pmod{m}$ iff (if and only if) $a \bmod m = b \bmod m$.

Example 2:
Determine whether 17 is congruent to 24 modulo 7 and whether 24 and 14 are congruent modulo 6.

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The questions posed: $17 \equiv 24 \pmod{7}$? Yes
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1) $17 \bmod 7 = 3$, and $24 \bmod 7 = 3$, therefore $17 \equiv 24 \pmod{7}$.

2) $24 \bmod 6 = 0$, and $14 \bmod 6 = 2$,

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Let *a,b* be integers, and *m* be a positive integer. Then $a \equiv b \pmod{m}$ iff (if and only if) $a \bmod m = b \bmod m$.

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List three integers that are congruent to *3* modulo *5*.

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how about negative ones? - don't forget that we need to be carefull
$(-2) \cdot 5 + 3 = -7$ check: $-7 = (-2) \cdot 5 + 3$ True
Answer: {13, 23, 8, -7}

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Example: assume that $13 = 1 + 3 \cdot 4$, then $13 \equiv 1 \pmod{4}$

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If $m \mid (a-b)$ then by **def.** of divisibility $\exists k \in \mathbf{Z}\ (a-b = k \cdot m)$

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$\exists k \in \mathbf{Z}\ (a = km + b)$

qed

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Let $m \in \mathbf{Z}^+$. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then
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Examples:
$5 \equiv 1 \pmod 4$ and $3 = 11 \pmod 4$, then

$5+3 \equiv 1+11 \pmod 4$, i.e. $8 \equiv 12 \pmod 4$

and

$5 \cdot 3 \equiv 1 \cdot 11 \pmod 4$, i.e. $15 \equiv 11 \pmod 4$

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$ac = (b+km)\,(d+sm) = bd + bsm + kdm + ksm^2 = bd + m(bs + kd + ksm)$

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[Corollary]
Let $m \in \mathbf{Z}^+$, and $a,b \in \mathbf{Z}$. Then
$(a+b) \bmod m = ((a \bmod m)+(b \bmod m)) \bmod m$
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Proof: let $a \equiv b \pmod{m}$, and $c \equiv d \pmod{m}$
then $\exists k \in \mathbf{Z}\ (a = b+km)$ and $\exists s \in \mathbf{Z}\ (c = d+sm)$

$a + c = b+d+km+sm$ or $(a + c) = (b+d)+m(k+s)$
therefore by **Theorem 4**, $a + c \equiv b + d \pmod{m}$, with integer $k' = (k+s)$

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Proof: not presented here

6.2 Modular arithmetic

Theorem 5 and its **corollary** are very useful in computing arithmetic expressions **mod** *n*

Example: to compute 15^{120} **mod** *68*

we would do 15^{120} first, and then take the result **mod** *68*.

-use the calculator to calculate it

15^{120} is a very large number,
multiplying large numbers is less efficient than multiplying small numbers

6.2 Modular arithmetic

Theorem 5 and its **corollary** are very useful in computing arithmetic expressions **mod** *n*

Example: to compute 15^{120} **mod** *68*

we would do 15^{120} first, and then take the result **mod** *68*.

-use the calculator to calculate it

<u>another approach</u>:

$15^{120} = \underbrace{15 \cdot 15 \cdot \ldots \cdot 15}_{120 \quad 15s}$

It is meaningless to do
((15 **mod** 68)(15 **mod** 68) … (15 **mod** 68))**mod** 68

[Theorem 5]
Let $m \in \mathbf{Z}^+$, and $a,b \in \mathbf{Z}$.
If $a \equiv b$ (***mod*** m) and
$c \equiv d$ (***mod*** m) then
$a + c \equiv b + d$ (***mod*** m), and
$ac \equiv bd$ (***mod*** m)
[Corollary]
$(a+b)$ ***mod*** $m = ((a$ ***mod*** $m)+(b$ ***mod*** $m))$ ***mod*** m
ab ***mod*** $m = ((a$ ***mod*** $m)(b$ ***mod*** $m))$ ***mod*** m

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It is meaningless to do
((15 **mod** 68)(15 **mod** 68) … (15 **mod** 68))**mod** 68

Instead do
p = (15 · 15) mod 68
p = (p · 15) mod 68 and so forth … 119 times

6.2 Modular arithmetic

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Example: to compute 15^{120} **mod** *68*

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-use the calculator to calculate it

<u>another approach</u>:

$15^{120} = \underbrace{15 \cdot 15 \cdot \ldots \cdot 15}_{120 \quad 15s}$

Instead do

p = (15 · 15) mod 68

p = (p · 15) mod 68 and so forth … 119 times

```
p := 15

For i:= 1 to 119
 p := (15*p) mod 68
End-for

Return(p)
```

i = 1 , p = 21
i = 2 , p = 43
i = 3 , p = 33
i = 4 , p = 19
i = 5 , p = 13
i = 6 , p = 59
i = 7 , p = 1
i = 8 , p = 15
i = 9 , p = 21
i = 10 , p = 43
i = 11 , p = 33
i = 12 , p = 19
i = 13 , p = 13
i = 14 , p = 59
i = 15 , p = 1
i = 16 , p = 15
i = 17 , p = 21
i = 18 , p = 43
i = 19 , p = 33
i = 20 , p = 19
i = 21 , p = 13
i = 22 , p = 59
i = 23 , p = 1
i = 24 , p = 15
i = 25 , p = 21
i = 26 , p = 43
i = 27 , p = 33

i = 28 , p = 19
i = 29 , p = 13
i = 30 , p = 59
i = 31 , p = 1
i = 32 , p = 15
i = 33 , p = 21
i = 34 , p = 43
i = 35 , p = 33
i = 36 , p = 19
i = 37 , p = 13
i = 38 , p = 59
i = 39 , p = 1
i = 40 , p = 15
i = 41 , p = 21
i = 42 , p = 43
i = 43 , p = 33
i = 44 , p = 19
i = 45 , p = 13
i = 46 , p = 59
i = 47 , p = 1
i = 48 , p = 15
i = 49 , p = 21
i = 50 , p = 43
i = 51 , p = 33
i = 52 , p = 19
i = 53 , p = 13
i = 54 , p = 59

i = 55 , p = 1
i = 56 , p = 15
i = 57 , p = 21
i = 58 , p = 43
i = 59 , p = 33
i = 60 , p = 19
i = 61 , p = 13
i = 62 , p = 59
i = 63 , p = 1
i = 64 , p = 15
i = 65 , p = 21
i = 66 , p = 43
i = 67 , p = 33
i = 68 , p = 19
i = 69 , p = 13
i = 70 , p = 59
i = 71 , p = 1
i = 72 , p = 15
i = 73 , p = 21
i = 74 , p = 43
i = 75 , p = 33
i = 76 , p = 19
i = 77 , p = 13
i = 78 , p = 59
i = 79 , p = 1
i = 80 , p = 15
i = 81 , p = 21

```
p := 15

For i:= 1 to 119
 p := (15*p) mod 68
End-for

Return(p)
```

i = 82 , p = 43
i = 83 , p = 33
i = 84 , p = 19
i = 85 , p = 13
i = 86 , p = 59
i = 87 , p = 1
i = 88 , p = 15
i = 89 , p = 21
i = 90 , p = 43
i = 91 , p = 33
i = 92 , p = 19
i = 93 , p = 13
i = 94 , p = 59
i = 95 , p = 1
i = 96 , p = 15
i = 97 , p = 21
i = 98 , p = 43
i = 99 , p = 33
i = 100 , p = 19

i = 101 , p = 13
i = 102 , p = 59
i = 103 , p = 1
i = 104 , p = 15
i = 105 , p = 21
i = 106 , p = 43
i = 107 , p = 33
i = 108 , p = 19
i = 109 , p = 13
i = 110 , p = 59
i = 111 , p = 1
i = 112 , p = 15
i = 113 , p = 21
i = 114 , p = 43
i = 115 , p = 33
i = 116 , p = 19
i = 117 , p = 13
i = 118 , p = 59
i = 119 , p = 1

$15^{120} \bmod 68 = 1$

6.2 Modular arithmetic

Example: compute $(15^{12} + 17 \cdot 23)$ **mod** 7

$$(15^{12} + 17 \cdot 23) \textbf{ mod } 7 =$$
$$= [(15 \textbf{ mod } 7)^{12} + (17 \textbf{ mod } 7) \cdot (23 \bmod 7)] \textbf{ mod } 7 =$$
$$= [\quad 1^{12} \quad + \quad 3 \quad \cdot \quad 2 \quad] \textbf{ mod } 7 =$$
$$= [\quad 1 \quad + \quad 6 \quad] \textbf{ mod } 7 = 7 \textbf{ mod } 7 = 0$$

6.2 Modular arithmetic - rings

The operation defined by adding two numbers and applying **mod** *n* to the result is called addition **mod** *n*.

The operation defined by multiplying two numbers and applying **mod** *n* to the result is called multiplication **mod** *n*.

The set {0, 1, 2,..., n-1} along with addition and multiplication **mod** *n* defines a closed mathematical system with *n* elements called a ring.

Denotation: Z_n

Many different kinds of rings are studied in abstract mathematics.

We will only be concerned with rings of the form Z_n for some integer *n>1*

6.2 Modular arithmetic - rings

Example: let's consider Z_8

elements of the Z_8: 0,1,2,3,4,5,6,7 operations: + (**mod** 8), $\times$ (**mod** 8)

$3 + 5$ in Z_8 is (3+5) **mod** 8 = 8 **mod** 8 = 0

3×5 in Z_8 is (3×5) **mod** 8 = 15 **mod** 8 = 7

6.2 Applications of Congruences : Hashing Functions

The central computer at an insurance company maintains records for each of its customers.
How can memory locations be assigned so that customer records can be retrieved quickly?
- use a suitably chosen hashing function

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- a hashing function ***h*** assigns memory location ***h(k)*** to the record that has ***k*** as its key.

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In practice many different hashing functions are used. One of them is $h(k) = k \bmod m$, m is the number of available memory locations

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Requirements to hashing functions:
- Hashing functions should be easily evaluated, so that files can be quickly located. $h(k) = k \bmod m$ meets the requirement.

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Requirements to hashing functions:
- Hashing functions should be easily evaluated, so that files can be quickly located. $h(k) = k \bmod m$ meets the requirement.
- Hashing functions should be onto, so that all memory locations are possible. $h(k) = k \bmod m$ is onto.

6.2 Applications of Congruences : Hashing Functions

Example 5:
Let $m = 102$, $s_1 = 111678907$, $s_2 = 192876903$, and $s3 = 111679615$
Assume that $h(k) = k$ **mod** m. Find $h(s_1)$, $h(s_2)$, $h(s_3)$.

6.2 Applications of Congruences : Hashing Functions

Example 5:
Let $m = 102$, $s_1 = 111678907$, $s_2 = 192876903$, and $s3 = 111679615$
Assume that $h(k) = k$ **mod** m. Find $h(s_1)$, $h(s_2)$, $h(s_3)$.

$h(111678907) = 111678907 \bmod 102 =$
$111678907 \div 102 = 1094891.186...$

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Let $m = 102$, $s_1 = 111678907$, $s_2 = 192876903$, and $s3 = 111679615$
Assume that $h(k) = k$ **mod** m. Find $h(s_1)$, $h(s_2)$, $h(s_3)$.

$h(111678907) = 111678907 \bmod 102 = 25$

$111678907 \div 102 = 1094891.186...$

therefore $111678907 - 102 \cdot 1094891 = 25$

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Let $m = 102$, $s_1 = 111678907$, $s_2 = 192876903$, and $s3 = 111679615$
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$h(111678907) = 111678907 \bmod 102 = 25$

$111678907 \div 102 = 1094891.186...$

therefore $111678907 - 102 \cdot 1094891 = 25$

$h(192876903) = 192876903 \bmod 102 =$

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$h(111678907) = 111678907 \bmod 102 = 25$

$111678907 \div 102 = 1094891.186...$

therefore $111678907 - 102 \cdot 1094891 = 25$

$h(192876903) = 192876903 \bmod 102 =$

$192876903 \div 102 = 1890950.029...$

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$h(111678907) = 111678907 \bmod 102 = 25$

$111678907 \div 102 = 1094891.186...$

therefore $111678907 - 102 \cdot 1094891 = 25$

$h(192876903) = 192876903 \bmod 102 = 3$

$192876903 \div 102 = 1890950.029...$

therefore $192876903 - 102 \cdot 1890950 = 3$

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Assume that $h(k) = k$ **mod** m. Find $h(s_1)$, $h(s_2)$, $h(s_3)$.

$h(111678907) = 111678907 \bmod 102 = 25$

$111678907 \div 102 = 1094891.186...$

therefore $111678907 - 102 \cdot 1094891 = 25$

$h(192876903) = 192876903 \bmod 102 = 3$

$192876903 \div 102 = 1890950.029...$

therefore $192876903 - 102 \cdot 1890950 = 3$

$h(111679615) = 111679621 \bmod 102 =$

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$h(111678907) = 111678907 \bmod 102 = 25$
$111678907 \div 102 = 1094891.186...$
therefore $111678907 - 102 \cdot 1094891 = 25$

$h(192876903) = 192876903 \bmod 102 = 3$
$192876903 \div 102 = 1890950.029...$
therefore $192876903 - 102 \cdot 1890950 = 3$

$h(111679615) = 111679621 \bmod 102 =$
$111679621 \div 102 = 1094898.186$

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Assume that $h(k) = k$ **mod** m. Find $h(s_1)$, $h(s_2)$, $h(s_3)$.

$h(111678907) = 111678907 \bmod 102 = 25$
$111678907 \div 102 = 1094891.186...$
therefore $111678907 - 102 \cdot 1094891 = 25$

$h(192876903) = 192876903 \bmod 102 = 3$
$192876903 \div 102 = 1890950.029...$
therefore $192876903 - 102 \cdot 1890950 = 3$

$h(111679615) = 111679621 \bmod 102 = 25$
$111679621 \div 102 = 1094898.186$
therefore $111679621 - 102 \cdot 1094898 = 25$

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Assume that $h(k) = k$ **mod** m. Find $h(s_1)$, $h(s_2)$, $h(s_3)$.

$h(111678907) = 111678907 \bmod 102 = 25$
$111678907 \div 102 = 1094891.186...$
therefore $111678907 - 102 \cdot 1094891 = 25$

$h(192876903) = 192876903 \bmod 102 = 3$
$192876903 \div 102 = 1890950.029...$
therefore $192876903 - 102 \cdot 1890950 = 3$

$h(111679615) = 111679621 \bmod 102 = 25$
$111679621 \div 102 = 1094898.186$
therefore $111679621 - 102 \cdot 1094898 = 25$

Answer: $h(s_1) = 25$, $h(s_2) = 3$, $h(s_3) = 25$.

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Assume that $h(k) = k$ **mod** m. Find $h(s_1)$, $h(s_2)$, $h(s_3)$.

$h(111678907) = 111678907 \bmod 102 = 25$
$111678907 \div 102 = 1094891.186...$
therefore $111678907 - 102 \cdot 1094891 = 25$

$h(192876903) = 192876903 \bmod 102 = 3$
$192876903 \div 102 = 1890950.029...$
therefore $192876903 - 102 \cdot 1890950 = 3$

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$111679621 \div 102 = 1094898.186$
therefore $111679621 - 102 \cdot 1094898 = 25$

Answer: $h(s_1) = 25$, $h(s_2) = 3$, $h(s_3) = 25$.

Did you notice that this hashing function is not one-to-one?

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Example 5:
Let $m = 102$, $s_1 = 111678907$, $s_2 = 192876903$, and $s3 = 111679621$
Assume that $h(k) = k$ **mod** m. Find $h(s_1)$, $h(s_2)$, $h(s_3)$.

$h(111678907) = 111678907 \bmod 102 = 25$
$111678907 \div 102 = 1094891.245...$
therefore $111678907 - 102 \cdot 1094891 = 25$

$h(192876903) = 192876903 \bmod 102 = 3$
$192876903 \div 102 = 1890950.029...$
therefore $192876903 - 102 \cdot 1890950 = 3$

$h(111679621) = 111679621 \bmod 102 = 25$
$111679621 \div 102 = 1094898.245...$
therefore $111679621 - 102 \cdot 1094898 = 25$

Answer: $h(s_1) = 25$, $h(s_2) = 3$, $h(s_3) = 25$.

Did you notice that this hashing function is not one-to-one?
- collision occurs when more than one file is assigned to the same memory location.

6.2 Applications of Congruences : Pseudo-random Numbers

randomly chosen numbers are often needed for computer simulations and other applications.

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Different methods have been devised for generating numbers that have properties close to randomly chosen numbers, but they are still not random numbers, because they were generated by *systematic methods*.

Such numbers are called pseudorandom numbers.

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Such numbers are called pseudorandom numbers.

One of the methods: linear congruential method
- we choose 4 integers:

the **modulus** m,	with
the **multiplier** a,	$2 \le a < m$,
the **increment** c, and	$0 \le c < m$, and
the **seed** x_0,	$0 \le x_0 < m$

Sequence $\{x_n\}$ of pseudorandom numbers is generated by

$x_{n+1} = (ax_n + c)$ **mod** m

$$0 \le x_n < m, \text{ for all } n$$

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Sequence $\{x_n\}$ of pseudorandom numbers is generated by

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$$0 \le x_n < m, \text{ for all } n$$

If $c = 0$, then the generator is called a pure multiplicative generator

6.2 Pseudo-random Numbers

modulus m, **multiplier** a, **increment** c, and the **seed** x_0,

$2 \le a < m$, $0 \le c < m$, and $0 \le x_0 < m$

Sequence $\{x_n\}$ of pseudorandom numbers is generated by

$x_{n+1} = (ax_n + c)$ **mod** m, where $0 \le x_n < m$, for all n

Example 6*:*
What sequence of pseudorandom numbers is generated using the linear congruential generator $x_{n+1} = (3x_n + 1)$ ***mod*** 11, with the seed $x_0 = 2$?

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Solution:

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$x_1 = (3 \cdot 2 + 1) \bmod 11 =$

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Solution:
$x_0 = 2$

$x_1 = (3 \cdot 2 + 1) \bmod 11 = 7 \bmod 11 = 7$

modulus m, **multiplier** a, **increment** c, and the **seed** x_0,

$2 \le a < m$, $0 \le c < m$, and $0 \le x_0 < m$

Sequence $\{x_n\}$ of pseudorandom numbers is generated by

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Solution:

$x_0 = 2$

$x_1 = (3 \cdot 2 + 1) \bmod 11 = 7 \bmod 11 = 7$

$x_2 = (3 \cdot 7 + 1) \bmod 11$

6.2 Pseudo-random Numbers

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Solution:

$x_0 = 2$

$x_1 = (3 \cdot 2 + 1) \bmod 11 = 7 \bmod 11 = 7$

$x_2 = (3 \cdot 7 + 1) \bmod 11 = 22 \bmod 11 = 0$

modulus m, **multiplier** a, **increment** c, and the **seed** x_0,

$2 \le a < m$, $0 \le c < m$, and $0 \le x_0 < m$

Sequence $\{x_n\}$ of pseudorandom numbers is generated by

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Solution:

$x_0 = 2$

$x_1 = (3 \cdot 2 + 1) \bmod 11 = 7 \bmod 11 = 7$

$x_2 = (3 \cdot 7 + 1) \bmod 11 = 22 \bmod 11 = 0$

$x_3 = (3 \cdot 0 + 1) \bmod 11 =$

modulus m, **multiplier** a, **increment** c, and the **seed** x_0,

$2 \le a < m$, $0 \le c < m$, and $0 \le x_0 < m$

Sequence $\{x_n\}$ of pseudorandom numbers is generated by

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What sequence of pseudorandom numbers is generated using the linear congruential generator $x_{n+1} = (3x_n + 1)$ ***mod*** 11, with the seed $x_0 = 2$?

Solution:

$x_0 = 2$

$x_1 = (3 \cdot 2 + 1) \bmod 11 = 7 \bmod 11 = 7$

$x_2 = (3 \cdot 7 + 1) \bmod 11 = 22 \bmod 11 = 0$

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6.2 Pseudo-random Numbers

modulus m, **multiplier** a, **increment** c, and the **seed** x_0,

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$x_6 = (3 \cdot 2 + 1) \bmod 11 = 7 \bmod 11 = 7$ – not good, because from now on we have a pattern in number-generation:

2 7 0 1 4 2 7 0 1 4 2 7 0 1 4 2 7 0 1 4 ....