Chapter 2. Basic Structures: Sets, Functions, Sequences, Sums

2.5 Set identities
2.6 Cartesian Products.

2.2 Set Identities

A set identity is an equation involving sets that is true regardless of the contents of the sets in the expression.

The idea is similar to an equivalence in logic which holds regardless of the truth values of the individual variables in the expressions.

The laws of propositional logic can be used to derive corresponding set identities.

Note that the sets U and ∅ correspond to the constants true (T) and false (F):

$$x \in \emptyset \leftrightarrow F \qquad\qquad x \in U \leftrightarrow T$$

Set Identities

Here is the list of the most important set identities.

Identity	***Name***
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $\quad$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $\quad$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\quad$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $\quad$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $\quad$ $A \cap \overline{A} = \emptyset$	Complement laws

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Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

<u>Proof</u>:

In order to prove that these two sets are equal, we need to show that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A \cup B} \supseteq \overline{A} \cap \overline{B}$.

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a) Let's prove that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$:
assume that $x \in \overline{A \cup B}$. It means (by definition) that $x \notin A \cup B$, which means that $x \notin A$ and $x \notin B$.
If $x \notin A$ and $x \notin B$, then $x \in \overline{A}$, and $x \in \overline{B}$ therefore (by definition of intersection of two sets) $x \in \overline{A} \cap \overline{B}$.

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If $x \in \overline{A}$, then $x \notin A$. If $x \in \overline{B}$, then $x \notin B$.
Therefore $x \in \overline{A}$ and $x \in \overline{B}$ implies that $x \notin A$ and $x \notin B$, hence $x \notin A \cup B$; which means that $x \in \overline{A \cup B}$.

qed

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Solution: three ways (use laws, use tables and prove $(A \cap B) \cup (A \cap \overline{B}) \subseteq A$ with $(A \cap B) \cup (A \cap \overline{B}) \supseteq A$)

$(A \cap B) \cup (A \cap \overline{B}) = A \cap (B \cup \overline{B})$ *by Distributive laws*

$= A \cap U$ *by Complement laws*

$= A$ *by Identity laws*

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Cartesian Products

The order of elements in the set is often important. Sets are unordered. Ordered n-tuples provide ordered collection.

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n-tuples are equal iff each corresponding pair of their elements is equal, i.e. $(a_1, a_2, ..., a_n)=(b_1, b_2, ..., b_n)$ *iff* $a_i=b_i$, for $i=1,2, ..., n$

Examples:
1) (1,2,3,4)
2) (a,b)
3) (1,f,2,g)

Cartesian Products

Cartesian product of two sets, A and B, is the set of all ordered pairs (a,b), where $a \in A$ and $b \in B$

denotation: $A \times B$

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$B \times A = \{(a,1),(a,2),(a,3),(a,4),(b,1),(b,2),(b,3),(b,4)\}$

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No

$A \times B = B \times A$ only if A=B

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