Chapter 2. Basic Structures: Sets, Functions, Sequences, Sums

2.1 Sets and subsets
2.2 Sets of sets

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- used to group objects together,
- often the objects with similar properties

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This description of a set (without specification what an object is) was started by the German mathematician Georg Cantor, in 1895.

He is considered to be the founder of **set theory**.

We will use Cantor's original version of set theory (called **naive set theory**).

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Example 1:

A={a,b,c,d,e} "set A contains elements a, b, c, d, and e"

$a \in A$ "a is an element of set A"
"a belongs to A"

$f \notin A$ "f is not an element of set A"
"f doesn't belong to A"

Example 2:

A={0, 1, 2, 3, ..., 100}
“set A contains elements integers from 0 to 100, including”

$77 \in A$ $120 \notin A$

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A = { x | x is even positive integer}

which means that A = { 2, 4, 6, 8, 10, ...}

-we can use this notation when it is not possible to list all the elements of the set.

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Example 4 (set builder notation):

T = { x | x^2=64} Can you tell what numbers set T consists of?
T={-8,8}

Some of the known sets

N = { 0, 1, 2, 3, 4, ...} set of natural numbers
(in math: **W** – set of whole numbers)

Z = {...,-3, -2, -1, 0, 1, 2, 3, ...} set of integers

$\mathbf{Z}^+$ = { 1, 2, 3, 4, ...} set of positive integers

R – the set of real numbers (rational and irrational numbers)

Q = $\{ \frac{p}{q} \mid p \in Z, q \in Z, q \neq 0\}$ set of fractions (quotients)

2.1 Sets and subsets.

Two sets are equal if and only if (iff) they have the same elements

A = B iff $\forall x\, (x \in A \leftrightarrow x \in B)$

Example 5:

a) {1, 4, 6, 7} = {1, 6, 7, 4} = {1, 1, 6, 7, 4, 4}
the order doesn't matter;
how many times element is listed doesn't matter either.

b) {0,{1},6} = {6, 0, 0, 6, {1},{1}}

c) {0,{1},6} $\neq$ {0, 1, 6}

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Example 6: For each of the following sets determine whether 2 is an element of that set.

a) { x ∈ R | x is an integer greater than 1 }
b) { x ∈ R | x is the square of an integer }
c) {2, {2}}
d) {{2},{{2}}}
e) {{2},{2,{2}}}
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2.1 Sets and subsets.

Venn Diagrams

- is used to show relationships between sets.
- named after English mathematician Jogn Venn, who introduced their use in 1881.

In Venn diagram the universal set *U*, which contains all the objects under consideration, is represented by a rectangle.

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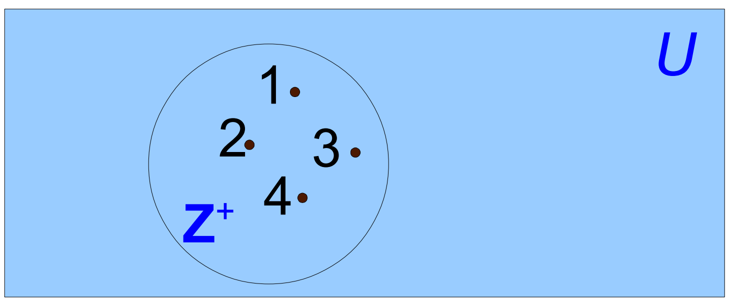


U – universe

Let *U* = **Z** the set of integers

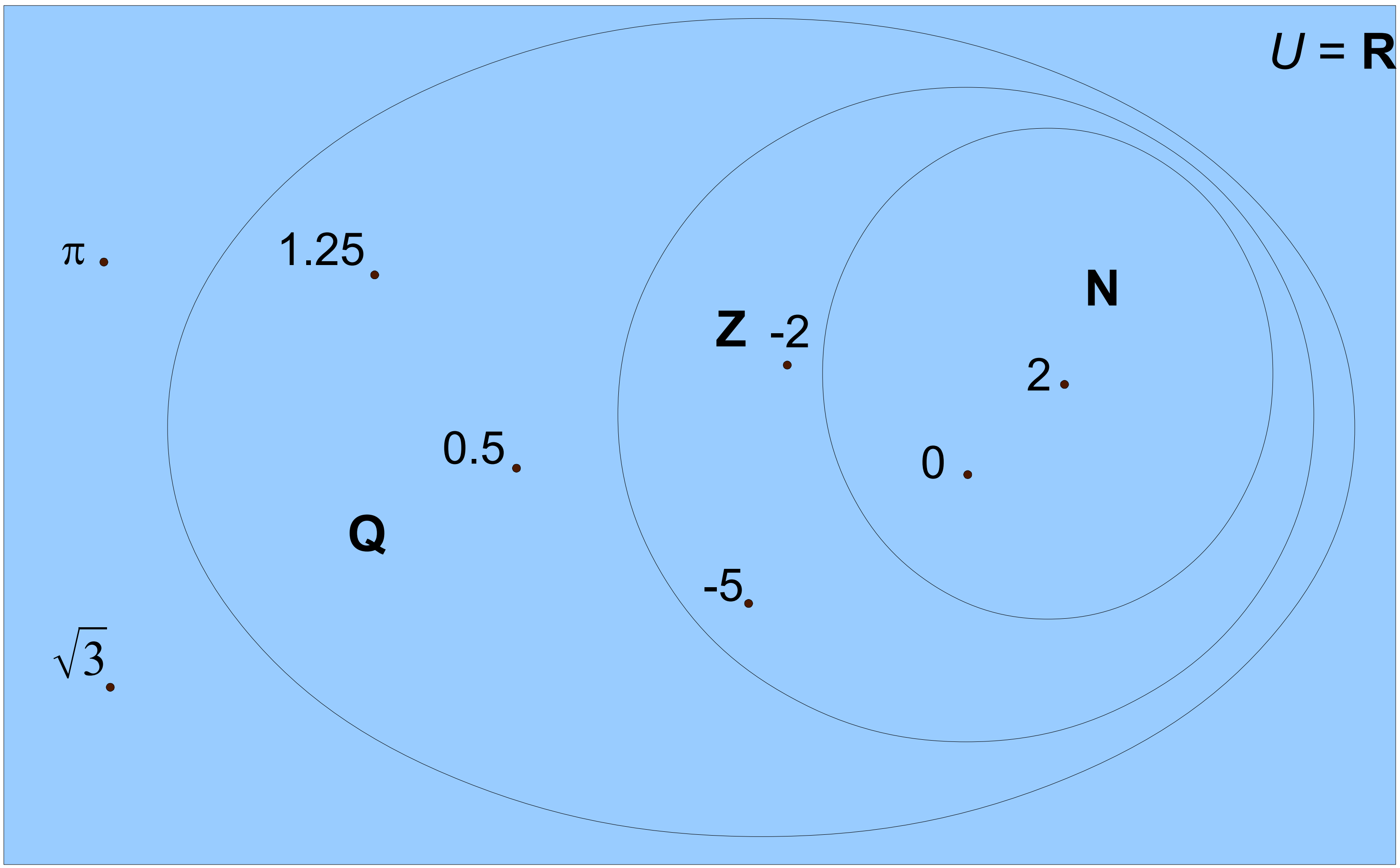
Inside the rectangle, circles and other geometrical figures are used to represent sets.

Sometimes points are used to represent particular elements of the set.

2.1 Sets and subsets.

Venn Diagrams

Venn diagrams allow us to visualize relationships between sets.



2.1 Sets and subsets.

Empty set is a set that has no elements; denotation: $\varnothing$

Singleton set is a set that has exactly one element.

Examples 7:

1) A={2} – singleton set

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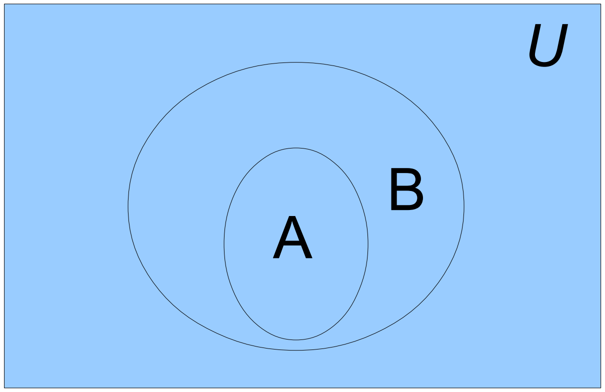
A is a subset of B iff every element of A is also an element of B.

denotation: $A \subseteq B$ or $B \supseteq A$

We can re-write the definition in Predicate Logic:
$A \subseteq B$ iff $\forall x(x \in A \rightarrow x \in B)$ is true



Venn diagram for $A \subseteq B$

Example 8:
Let A={1,2,3} and B={0,1,2,5,3}.
Then $A \subseteq B$.

2.1 Sets and subsets.

A is a proper subset of B iff A is a subset of B and set A and B are not equal.

<u>denotation</u>: $A \subset B$ or $B \supset A$

We can re-write the definition in Predicate Logic:
$A \subset B$ iff $\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$ is true

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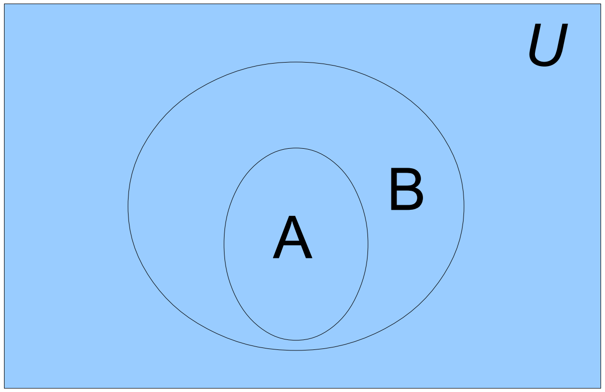
2) Let A={1,2,3} and B={1,2,3}
Then $A \subseteq B$, but $A \not\subset B$



Venn diagram for $A \subseteq B$

Theorem 1

For every set S,

(1) $\emptyset \subseteq S$

(2) $S \subseteq S$

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Proof:

(1) Let S be a set.

To show that $\varnothing \subseteq S$, we need to show that $\forall x(x \in \varnothing \to x \in S)$ is true: empty set contains no elements, therefore, $x \in \varnothing$ is false for any x; then $x \in \varnothing \to x \in S$ is true for any x,because the precedent is false; hence $\forall x(x \in \varnothing \to x \in S)$ is true. We proved that $\varnothing \subseteq S$ for every set S.

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To show that $S \subseteq S$, we need to show that $\forall x(x \in S \rightarrow x \in S)$ is true: implication $x \in S \rightarrow x \in S$ is true for any x ($p \rightarrow p$ is a tautology); hence $\forall x(x \in S \rightarrow x \in S)$ is true.

qed

2.1 Sets and subsets.

Let S be a set. If there are exactly *n* distinct elements in S, where *n* is a nonnegative integer, we say that S is a finite set and that *n* is the cardinality of S.

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a) Let A be a set of even positive integers less than 8.

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a) Let A be a set of even positive integers less than 8.
|A| = 3, because A = {2,4,6}

b) Let B be a set of letters in English alphabet.
|B|=26, because B = {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z}

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c) S = $\emptyset$
|S|=0

A set is infinite if it is not finite.

Examples 11:

a) **N** = {0, 1, 2, 3, 4, 5, ...} |**N**| = ∞

b) S={ x | x$\in$N and x is even} |S| = ∞

infinity

2.2 Sets of sets

It is possible that the elements of a set are themselves sets.
For example, consider the set B:

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Given a set S, the power set of S is the set of all subsets of the set S.

denotation: P(S)

Examples 12: find the power set of the given sets

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a) A = {1,2,3}
P(A) = P({1,2,3}) = {∅,{1},{2},{3},{1,2},{2,3},{1,3},{1,2,3}}

note: empty set and the set A itself are the subsets of A.

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P(S) = {∅,{∅}}

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b) B = ∅

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c) S = {∅}

P(S) = {∅,{∅}}

Did you notice that power set is a set of sets?

a possible mistake:

P({1,2,3}) = {∅,1,2,3,{1},{2},{3},{1,2},{2,3},{1,3},{1,2,3}}

2.2 Sets of sets

Theorem

Given a set S, with n elements, the power set of S has 2^n elements, i.e. $|P(s)|=2^n$

recall Example 12(a):
A = {1,2,3},
P(A) = P({1,2,3}) = {∅,{1},{2},{3},{1,2},{2,3},{1,3},{1,2,3}}

$|A| = 3$, $|P(A)| = 8 = 2^3$