Chapter 1. The Foundations: Logic and Proofs

1.9-1.10 Nested Quantifiers

Nested Quantifiers

Two quantifiers are nested if one is within the scope of the other.

Recall one of the examples from the previous class: $\forall x \,(P(x) \vee \exists y\, Q(x,y))$

Let's learn to “read” the quantified predicates. **Example 1**:

a) $\forall x\, \exists y\, (x+y = 0)$ - *additive inverse*

we can re-write it as $\forall x Q(x)$, where $Q(x)$ is $\exists y\, (x+y = 0)$

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Let the domain be all real numbers.

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b) $\forall x\ \forall y\ (\ x+y = y+x\ )$ - *commutativity of addition*

Reading: “For any x and y, x+y is equal to y+x”

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Reading: “For any x and y, x+y is equal to y+x”

c) $\forall x\ \forall y\ ((x>0) \wedge (y<0) \rightarrow (xy<0))$

Reading: “For any x and y, if x is greater than 0 and y is less than 0, then their product is less than 0”

Order of quantifications: from the left to the right

Nested Quantifiers

Example 2: Let $Q(x,y)$ denote "$x-y=0$". What are the truth values of the quantifications $\exists y\ \forall x\ Q(x,y)$ and $\forall x\ \exists y\ Q(x,y)$.

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Example 2: Let Q(x,y) denote "x-y=0". What are the truth values of the quantifications ∃y ∀x Q(x,y) and ∀x ∃y Q(x,y).

Reasoning:

1) ∃y ∀x Q(x,y)

Let's read it: "there exists a value of *y*, such that for any *x*, *x-y=0*".

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1) $\exists y\ \forall x\ Q(x,y)$

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1) $\exists y\ \forall x\ Q(x,y)$

Let's read it: "there exists a value of *y*, such that for any *x*, *x-y=0*".
It is False, because for a fixed *y*, there is only one x value, such that x-y=0

2) $\forall x\ \exists y\ Q(x,y)$

Let's read it: "for any *x*, there exists *y*, such that *x-y=0*"
It is True, because when we take any value of x (from now on it is fixed) we can find a value for y: y=x which will give us that x-y=0.

Nested Quantifiers

on page 60 in the book, there is a nice table which shows quantifications of two variables.

Statement	When True?	When False?
$\forall y\ \forall x\ P(x,y)$ $\forall x\ \forall y\ P(x,y)$	P(x,y) is true for any pair x, y	there is a pair x, y for which P(x,y) is False
$\forall x\ \exists y\ P(x,y)$	For any x, there is a y, for which P(x,y) s true	There is an x, for which P(x,y) is false for any y
$\exists x\ \forall y\ P(x,y)$	There exists an x, for which P(x,y) is true for any y	There is no x, such that P(x,y) is true for any y
$\exists x\ \exists y\ P(x,y)$ $\exists y\ \exists x\ P(x,y)$	There is a pair x, y for which P(x,y) is true	P(x,y) is false for every pair x, y

Nested Quantifiers

Now, let's **translate mathematical statements into statements with nested quantifiers**:

Example 3:
Express each of these statements using predicates, quantifiers, logical connectives, and mathematical operators where domain consists of all integers.

a) The product of two negative integers is positive.

c) The difference of two negative integers is not necessarily negative.

d) The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers.

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we prefer this writing

$\forall x<0\ \forall y<0\ (x*y>0)$ or $\forall x \forall y\ (\ (x<0) \wedge (y<0) \rightarrow x*y>0\ )$

restricted quantifiers

c) The difference of two negative integers is not necessarily negative.

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$\forall x \forall y\ (\ (x<0) \land (y<0) \rightarrow x * y>0\ )$

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c) The difference of two negative integers is not necessarily negative.

$\exists x \exists y\ (\ (x<0) \land (y<0) \land (x-y) \geq 0\ )$

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$\forall x \forall y\ (\ |x+y| \leq |x|+|y|\ )$

Now, let's **translate from statements with nested quantifiers into English**:

Example 4:
Let T(x,y) mean "student x likes cuisine y". Where the domain for x consists of all students at your school and the domain for y consists of all cuisines. Express each of these statements by a simple English sentence.

d) $\forall x \forall z \exists y \, ((x \neq z) \rightarrow \neg(T(x,y) \wedge T(z,y))$

e) $\exists x \exists z \forall y \, (T(x,y) \leftrightarrow T(z,y))$

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Let's re-write it a little bit: $\forall x \forall z \exists y\ ((x \neq z) \rightarrow (\neg T(x,y) \vee \neg T(z,y))$
“For any two different students there exists a cuisine such that at least one of them dislikes it.”

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e) $\exists x \exists z \forall y\ (\ T(x,y) \leftrightarrow T(z,y))$

"There is a pair of students with the same tastes in cuisines: they like/dislike the same cuisines."

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Now, let's **translate English sentences into Logical Expressions**:

Example 5:
Let F(x,y) be the statement "x can fool y", where the domain for both x and y consists of all people in the world. Use quantifiers to express each of these statements.

a) Everybody can fool Fred.

d) There is no one who can fool everybody.

h) There is exactly one person whom everybody can fool.

j) There is someone who can fool exactly one person besides himself or herself.

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$\exists x \exists! y(y \neq x \wedge F(x,y))$

Negating Nested Quantifiers

Example 6:
Negate the given statements, then re-write them so that negations appear only within predicates (i.e. no negation is outside a quantifier or an expression involving logical connectives)

a) $\forall x \exists y\ (x * y = 3)$

b) $\exists x \forall a \exists y\ (F(x,y) \wedge A(y,a))$

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$\neg\ \forall x \exists y\ (x*y=3) \equiv \exists x \forall y\ \neg(x*y=3) = \exists x \forall y\ (x*y \neq 3)$

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b) $\exists x \forall a \exists y\ (F(x,y) \land A(y,a))$

$\neg\ \exists x \forall a \exists y\ (F(x,y) \land A(y,a)) \equiv \forall x \exists a \forall y\ \neg(F(x,y) \land A(y,a))$

$\equiv\ \forall x \exists a \forall y\ (\neg F(x,y) \lor \neg A(y,a))$