Chapter 1. The Foundations: Logic and Proofs

1.4 Logical Equivalences
1.5 Laws of Propositional Logic

Propositional Equivalences

An important step used in a mathematical argument is the replacement of a statement with another statement with the same truth value.

tautology is a compound proposition that is always true, no matter what the truth values of the propositions that occur in it.

contradiction is a compound proposition that always false.

contingency is a compound statement that is neither a tautology nor a contradiction.

Propositional Equivalences

An important step used in a mathematical argument is the replacement of a statement with another statement with the same truth value.

tautology is a compound proposition that is always true, no matter what the truth values of the propositions that occur in it.

example: $\neg p \vee p$

p	$\neg p$	$\neg p \vee p$
0	1	1
1	0	1

contradiction is a compound proposition that always false.

example: $\neg p \wedge p$

p	$\neg p$	$\neg p \wedge p$
0	1	0
1	0	0

contingency is a compound statement that is neither a tautology nor a contradiction.

example: $p \rightarrow q \vee \neg p$

Propositional Equivalences

Two compound propositions, **p** and **q**, are *logically equivalent* if $p \leftrightarrow q$ is a tautology.

We'll write $p \equiv q$ or $p \Leftrightarrow q$

Ways to determine whether two compound proposition are equivalent:

- truth tables (columns giving their truth values agree)
- use laws

Example 1: proof by truth tables that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

p	q	$\neg p$	$p \rightarrow q$	$\neg p \vee q$
T	T	F		
T	F	F		
F	T	T		
F	F	T		

Propositional Equivalences

Two compound propositions, **p** and **q**, are *logically equivalent* if $p \leftrightarrow q$ is a tautology.

We'll write $p \equiv q$ or $p \Leftrightarrow q$

Ways to determine whether two compound proposition are equivalent:

- truth tables (columns giving their truth values agree)
- use laws

Example 1: proof by truth tables that $p \to q$ and $\neg p \vee q$ are logically equivalent.

p	q	$\neg p$	$p \to q$	$\neg p \vee q$
T	T	F	T	
T	F	F	F	
F	T	T	T	
F	F	T	T	

Propositional Equivalences

Two compound propositions, **p** and **q**, are *logically equivalent* if $p \leftrightarrow q$ is a tautology.

We'll write $p \equiv q$ or $p \Leftrightarrow q$

<u>Ways to determine whether two compound proposition are equivalent</u>:

- truth tables (columns giving their truth values agree)
- use laws

Example 1: proof by truth tables that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

p	q	¬p	$p \rightarrow q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Propositional Equivalences

Two compound propositions, **p** and **q**, are *logically equivalent* if $p \leftrightarrow q$ is a tautology.

We'll write $p \equiv q$ or $p \Leftrightarrow q$

Ways to determine whether two compound proposition are equivalent:

- truth tables (columns giving their truth values agree)
- use laws

Example 1: proof by truth tables that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Reading of both compound propositions:

Let p: "the weather is good" and q: "we'll go swimming" Then,

$p \rightarrow q$: "If the weather is good, we'll go swimming"; and

$\neg p \vee q$: "the weather is not good or we'll go swimming"

Laws of Propositional Logic

Here is a list of some important equalities:

$\neg(p \vee q) \equiv \neg p \wedge \neg q$ $\neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's Laws (1)	$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws (5)
$\neg(\neg p) \equiv p$	Double negation law (2)	$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws (6)
$p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$	Commutative laws (3)	$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws (7)
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws (4)	$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws (8)
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$		Associative laws (9)	
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$		Distributive laws (10)	

Laws of Propositional Logic

Here is more of important equalities:

$p \to q \equiv \neg p \lor q$	(11)	$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$	(12)
$p \to q \equiv \neg q \to \neg p$	(13)	$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$	(14)
$p \lor q \equiv \neg p \to q$	(15)	$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$	(16)
$p \land q \equiv \neg(p \to \neg q)$	(17)	$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$	(18)
$\neg(p \to q) \equiv p \land \neg q$	(19)		
$(p \to q) \land (p \to r) \equiv p \to (q \land r)$	(20)		
$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$	(21)		
$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$	(22)		
$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$	(23)		

Laws of Propositional Logic

Example 2: Show that $p \to q$ and $\neg q \to \neg p$ are logically equivalent using the above laws.

Laws of Propositional Logic

Example 2: Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent using the above laws.

We'll start from $\neg q \rightarrow \neg p$:

Example 2: Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent using the above laws.

We'll start from $\neg q \rightarrow \neg p$:

$\neg q \rightarrow \neg p \equiv \neg\neg q \vee \neg p$ by (11) $p \rightarrow q \equiv \neg p \vee q$

Propositional Equivalences

Example 2: Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent using the above laws.

We'll start from $\neg q \rightarrow \neg p$:

$\neg q \rightarrow \neg p \equiv \neg\neg q \vee \neg p$ by (11)

$\equiv q \vee \neg p$ by double negation law (2) $\neg(\neg p) \equiv p$

Example 2: Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent using the above laws.

We'll start from $\neg q \rightarrow \neg p$:

$\neg q \rightarrow \neg p \equiv \neg\neg q \vee \neg p$	by (11)	
$\equiv q \vee \neg p$	by double negation law (2)	
$\equiv \neg p \vee q$	by commutative laws (3)	$p \vee q \equiv q \vee p$

Example 2: Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent using the above laws.

We'll start from $\neg q \rightarrow \neg p$:

$\neg q \rightarrow \neg p \equiv \neg\neg q \vee \neg p$	by (11)
$\equiv q \vee \neg p$	by double negation law (2)
$\equiv \neg p \vee q$	by commutative laws (3)
$\equiv p \rightarrow q$	by (11) $p \rightarrow q \equiv \neg p \vee q$

Example 2: Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent using the above laws.

We'll start from $\neg q \rightarrow \neg p$:

$\neg q \rightarrow \neg p$	$\equiv \neg\neg q \vee \neg p$	by (11)
	$\equiv q \vee \neg p$	by double negation law (2)
	$\equiv \neg p \vee q$	by commutative laws (3)
	$\equiv p \rightarrow q$	by (11)

Example 3: Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent.

Example 2: Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent using the above laws.

We'll start from $\neg q \rightarrow \neg p$:

$\neg q \rightarrow \neg p \equiv \neg\neg q \vee \neg p$	by (11)
$\equiv q \vee \neg p$	by double negation law (2)
$\equiv \neg p \vee q$	by commutative laws (3)
$\equiv p \rightarrow q$	by (11)

Example 3: Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent.

$\neg(p \leftrightarrow q) \equiv$

$p \leftrightarrow \neg q \equiv$

Example 2: Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent using the above laws.

We'll start from $\neg q \rightarrow \neg p$:

$\neg q \rightarrow \neg p \equiv \neg\neg q \vee \neg p$	by (11)
$\equiv q \vee \neg p$	by double negation law (2)
$\equiv \neg p \vee q$	by commutative laws (3)
$\equiv p \rightarrow q$	by (11)

Example 3: Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent.

$\neg(p \leftrightarrow q) \equiv$

$\equiv \neg((p \rightarrow q) \wedge (q \rightarrow p))$ *by (12)*

$p \leftrightarrow \neg q \equiv$

$\equiv (p \rightarrow \neg q) \wedge (\neg q \rightarrow p)$ *by (12)*

Example 2: Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent using the above laws.

We'll start from $\neg q \rightarrow \neg p$:

$\neg q \rightarrow \neg p \equiv \neg\neg q \lor \neg p$ by (11)

$\equiv q \lor \neg p$ by double negation law (2)

$\equiv \neg p \lor q$ by commutative laws (3)

$\equiv p \rightarrow q$ by (11)

Example 3: Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent.

$\neg(p \leftrightarrow q) \equiv$

$\equiv \neg((p \rightarrow q) \land (q \rightarrow p))$ *by (12)*

$\equiv \neg(p \rightarrow q) \lor \neg(q \rightarrow p)$ *by (1)*

$\equiv \neg(\neg p \lor q) \lor \neg(\neg q \lor p)$ *by (11)*

$\equiv (\neg\neg p \land \neg q) \lor (\neg\neg q \land \neg p)$ *(1)*

$\equiv (p \land \neg q) \lor (q \land \neg p)$ *by (2)*

$p \leftrightarrow \neg q \equiv$

$\equiv (p \rightarrow \neg q) \land (\neg q \rightarrow p)$ *by (12)*

Propositional Equivalences

Example 2: Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent using the above laws.

We'll start from $\neg q \rightarrow \neg p$:

$\neg q \rightarrow \neg p \equiv \neg\neg q \vee \neg p$ by (11)

$\equiv q \vee \neg p$ by double negation law (2)

$\equiv \neg p \vee q$ by commutative laws (3)

$\equiv p \rightarrow q$ by (11)

Example 3: Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent.

$\neg(p \leftrightarrow q) \equiv$

$\equiv \neg((p \rightarrow q) \wedge (q \rightarrow p))$ *by (12)*

$\equiv \neg(p \rightarrow q) \vee \neg(q \rightarrow p)$ *by (1)*

$\equiv \neg(\neg p \vee q) \vee \neg(\neg q \vee p)$ *by (11)*

$\equiv (\neg\neg p \wedge \neg q) \vee (\neg\neg q \wedge \neg p)$ *(1)*

$\equiv (p \wedge \neg q) \vee (q \wedge \neg p)$ *by (2)*

$p \leftrightarrow \neg q \equiv$

$\equiv (p \rightarrow \neg q) \wedge (\neg q \rightarrow p)$ *by (12)*

$\equiv (\neg p \vee \neg q) \wedge (\neg\neg q \vee p)$ *by (11)*

$\equiv (\neg p \vee \neg q) \wedge (q \vee p)$ *by (2)*

$\equiv ((\neg p \vee \neg q) \wedge q) \vee ((\neg p \vee \neg q) \wedge p)$ *by (10)*

$\equiv (\neg p \wedge q) \vee (\neg q \wedge q) \vee (\neg p \wedge p) \vee (\neg q \wedge p)$

$\equiv (\neg p \wedge q) \vee \mathbf{F} \vee \mathbf{F} \vee (\neg q \wedge p)$ *by (8)*

$\equiv (\neg p \wedge q) \vee (\neg q \wedge p)$ *by (7)*

Propositional Equivalences

Example 4: Show that $((p \to q) \wedge (q \to r)) \to (p \to r)$ is a tautology without using truth tables.

Let's prove by contrapositive (see lecture 1 slides, slide 12)
(having $p \to q$, $\neg q \to \neg p$)

$\neg(p \to r) \to \neg((p \to q) \wedge (q \to r)) \equiv \neg\neg(p \to r) \vee \neg((p \to q) \wedge (q \to r))$ *by (11)*

$\equiv (p \to r) \vee (\neg(p \to q) \vee \neg(q \to r))$ *by double negation law (2)*

$\equiv (p \to r) \vee (\neg(p \to q) \vee \neg(q \to r))$ *by (1)*

$\equiv (\neg p \vee r) \vee (\neg(\neg p \vee q) \vee \neg(\neg q \vee r))$ *by (11) three times*

$\equiv (\neg p \vee r) \vee ((\neg\neg p \wedge \neg q) \vee (\neg\neg q \wedge \neg r))$ *by (1) twice*

$\equiv (\neg p \vee r) \vee ((p \wedge \neg q) \vee (q \wedge \neg r))$ *by double negation law (2) twice*

$\equiv \neg p \vee r \vee (p \wedge \neg q) \vee (q \wedge \neg r)$ *re-writing*

$\equiv (\neg p \vee (p \wedge \neg q)) \vee (r \vee (q \wedge \neg r))$ *re-grouping by (3)*

$\equiv ((\neg p \vee p) \wedge (\neg p \vee \neg q)) \vee ((r \vee q) \wedge (r \vee \neg r))$ *by (10)*

$\equiv (\mathbf{T} \wedge (\neg p \vee \neg q)) \vee ((r \vee q) \wedge \mathbf{T})$ *by (8) twice*

$\equiv (\neg p \vee \neg q) \vee (r \vee q)$ by (7) twice

$\equiv \neg p \vee \neg q \vee r \vee q \equiv \neg p \vee r \vee \mathbf{T}$ *by (8)*

$\equiv \mathbf{T}$ *by (6)*