Chapter 1. The Foundations: Logic and Proofs

1.11 Logical reasoning
1.12 Rules of inference with propositions

1.12 Rules of Inference for propositional logic

Let's go back to Propositional Logic.

In order to prove anything in mathematics, we need to present **valid arguments**.

argument is a sequence of statements that end with a conclusion.

valid means that the conclusion (or final statement) of the argument *must follow from the truth* of preceding statements, or **premises**, of the argument.

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valid means that the conclusion (or final statement) of the argument *must follow from the truth* of preceding statements, or **premises**, of the argument.

In other words: the *argument* with
premises $p_1, p_2, \ldots, p_n$ and *conclusion* q *is valid*, if
$(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$ is a tautology.

1.12 Rules of Inference for propositional logic

To deduce new statements from statements we already have, we use rules of inference, which are *templates* for constructing valid arguments.

rules of inference are our basic tools for establishing the truth of statements.

Fallacies are some common forms of incorrect reasoning, which lead to invalid arguments.

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p] \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$[(p) \wedge (q)] \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$	Resolution

1.12 Rules of Inference for propositional logic

Example 1 (use of Modus Ponens / MP):
Suppose that we have the conditional statement: "If it is hot today, then we will go swimming" and its hypothesis, "it is hot today".
Then by Modus Ponens, the conclusion of the conditional statement, "we will go swimming" is true.

p
$p \rightarrow q$
-------- *Modus Ponens*
q

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Then by Modus Ponens, the conclusion of the conditional statement, "we will go swimming" is true.

it is hot today
it is hot today $\rightarrow$ we will go swimming

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it is hot today	p	
it is hot today → we will go swimming	$p \rightarrow q$	
--	--------	*Modus Ponens*
we will go swimming	q	

Example 2: Determine whether the argument given here is valid and determine whether the conclusion must be true because of the validity of the argument.

“If $\sqrt{2} > \frac{3}{2}$, then $(\sqrt{2})^2 > (\frac{3}{2})^2$. We know that $\sqrt{2} > \frac{3}{2}$.

Consequently, $(\sqrt{2})^2 = 2 > (\frac{3}{2})^2 = \frac{9}{4}$ “.

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Consequently, $(\sqrt{2})^2 = 2 > (\frac{3}{2})^2 = \frac{9}{4}$ ".

$$\sqrt{2} > \frac{3}{2} \rightarrow (\sqrt{2})^2 > (\frac{3}{2})^2$$

$$\sqrt{2} > \frac{3}{2}$$

$$2 > \frac{9}{4}$$

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$$\sqrt{2} > \frac{3}{2} \longrightarrow (\sqrt{2})^2 > \left(\frac{3}{2}\right)^2$$

$$\sqrt{2} > \frac{3}{2}$$

$$2 > \frac{9}{4}$$

2 > 2.25 is false!

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$p \to q$
-------- *Modus Ponens*
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$\sqrt{2} > \frac{3}{2} \rightarrow (\sqrt{2})^2 > (\frac{3}{2})^2$

$\sqrt{2} > \frac{3}{2}$ *false* $\sqrt{2} = 1.414113\ldots$

$2 > \frac{9}{4}$ 2 > 2.25 is false!

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false: $\sqrt{2} > \frac{3}{2}$ → false: $(\sqrt{2})^2 > (\frac{3}{2})^2$	true	
$\sqrt{2} > \frac{3}{2}$	false	$\sqrt{2} = 1.414113\ldots$
$2 > \frac{9}{4}$	2 > 2.25 is false!	

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false $\sqrt{2} > \frac{3}{2}$	$\rightarrow$	*false* $(\sqrt{2})^2 > (\frac{3}{2})^2$	*true*
$\sqrt{2} > \frac{3}{2}$		*false*	$\sqrt{2} = 1.414113\ldots$
$2 > \frac{9}{4}$		2 > 2.25 is false!	

p

p → q

-------- *Modus Ponens*

q

Conclusion:

The **argument is valid**, because it is constructed by using MP, but **one of the premises is false**. Therefore we cannot conclude that the conclusion is True. Hence it is not necessary for the conclusion to be true if the argument is valid.

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a) Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.

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p

$p \vee q$

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p

------- *addition*

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a) Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.

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b) Jerry is a computer science major and a mathematics major. Therefore, Jerry is a mathematics major.

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b) Jerry is a computer science major and a mathematics major. Therefore, Jerry is a mathematics major.

$p \wedge q$

----

q

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b) Jerry is a computer science major and a mathematics major. Therefore, Jerry is a mathematics major.

$p \wedge q$

------- *simplification*

q

Example 3: What rule of inference is used in each of these arguments?

a) Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.

p

------- *addition*

$p \lor q$

b) Jerry is a computer science major and a mathematics major. Therefore, Jerry is a mathematics major.

$p \land q$

------- *simplification*

q

c) If it is snows today, we will go snow tubing. We didn't go snow tubing. Therefore, it didn't snow today.

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------- *simplification*

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$p \rightarrow q$

$\neg q$

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------- *simplification*

q

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$p \rightarrow q$

$\neg q$

------- *Modus tollens*

$\neg p$

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d) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.

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$p \to q$
$q \to r$

$p \to r$

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$p \to q$
$q \to r$
------- *hypothetical syllogism*
$p \to r$

Example 4: Use rules of inference to show that the hypotheses “Randy works hard,” “If Randy works hard, then he is a dull boy,” and “If Randy is a dull boy, then he will not get the job” imply the conclusion “Randy will not get the job.”

Using Rules of Inference to Build Arguments

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1. p
2. $p \rightarrow q$
3. $q \rightarrow r$

Using Rules of Inference to Build Arguments

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1. p
2. $p \to q$
3. $q \to r$

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4. q from 1 and 2 by MP

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3. $q \rightarrow r$

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4. q from 1 and 2 by MP
5. r from 3 and 4 by MP

Resolution

Computer programs have been developed to automate the task of reasoning and proving theorems. Many of these programs use resolution rule.

$p \vee q$

$\neg p \vee r$

--------- *Resolution* which is based on tautology $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$

$q \vee r$ ← resolvent

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Example 5: use resolution to show that the hypotheses “Jasmine is skiing or it is not snowing” and “It is snowing or Bart is playing hockey” imply that “Jasmine is skiing or Bart is playing hockey”

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$$\begin{array}{l} p \lor q \\ \neg p \lor r \\ \hline q \lor r \end{array}$$

--------- *Resolution* which is based on tautology $((p \lor q) \land (\neg p \lor r)) \rightarrow (q \lor r)$

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$p \vee r$

$q \vee r$

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$p \vee r$

----------- *resolution*

$q \vee r$

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$q \vee r$

(resolvent)

Example 6: Show that the hypotheses $(p \wedge q) \vee r$ and $r \rightarrow s$ imply the conclusion $p \vee s$.

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$q \vee r$

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1. $(p \wedge q) \vee r$
2. $r \rightarrow s$

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1. $(p \wedge q) \vee r$
2. $r \rightarrow s$

----

3. $\neg r \vee s$ re-writing 2 by law (11)

Resolution

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$p \vee q$
$\neg p \vee r$
--------- *Resolution* which is based $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$ on tautology
$q \vee r$

resolvent

Example 6: Show that the hypotheses $(p \wedge q) \vee r$ and $r \rightarrow s$ imply the conclusion $p \vee s$.

1. $(p \wedge q) \vee r$
2. $r \rightarrow s$

----

3. $\neg r \vee s$ re-writing 2 by law (11)
4. $(p \vee r) \wedge (q \vee r)$ re-writing 1 by (10)

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$p \vee q$
$\neg p \vee r$
--------- *Resolution* which is based on tautology $((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$
$q \vee r$

↑ resolvent

Example 6: Show that the hypotheses $(p \wedge q) \vee r$ and $r \to s$ imply the conclusion $p \vee s$.

1. $(p \wedge q) \vee r$
2. $r \to s$

----

3. $\neg r \vee s$ re-writing 2 by law (11)
4. $(p \vee r) \wedge (q \vee r)$ re-writing 1 by (10)
5. $(p \vee r)$ from 4 by simplification

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1. $(p \wedge q) \vee r$
2. $r \rightarrow s$

----

3. $\neg r \vee s$ re-writing 2 by law (11)
4. $(p \vee r) \wedge (q \vee r)$ re-writing 1 by (10)
5. $(p \vee r)$ from 4 by simplification
6. $p \vee s$ from 3 & 5 by resolution

Fallacies

fallacy of affirming the conclusion

fallacy of denying the hypothesis

Example 7:
If n is a real number, such that $n > 1$, then $n^2 > 1$.

Suppose that $n^2 > 1$, then $n > 1$.

Example 8:
If n is a real number with $n > 2$, then $n^2 > 4$.

Suppose $n \leq 2$, then ,$n^2 \leq 4$

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$p \rightarrow q$
q
------- not an inference rule
p

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$p \to q$
$\neg p$
------- not an inference rule
$\neg q$

Using a truth table to establish the validity of an argument

Example 9: consider the argument below

$p \vee q$

$\neg p$

----

q

The truth table:

p	q	¬p	p ∨ q
T	T		
T	F		
F	T		
F	F		

Using a truth table to establish the validity of an argument

Example 9: consider the argument below

$p \vee q$
$\neg p$
----
q

The truth table:

p	q	¬p	p ∨ q
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

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The truth table:

p	q	$\neg p$	$p \vee q$
T	T	F	T
T	F	F	T
F	**T**	**T**	**T**
F	F	T	F

Shows that there is only one case when both hypotheses are true, and in this case the conclusion is true as well.

Hence *the argument is valid*

Using a truth table to establish the invalidity of an argument

Example 10: consider the argument below

$p \rightarrow q$
q
----
p

The truth table:

p	q	$p \rightarrow q$
T	T	
T	F	
F	T	
F	F	

Using a truth table to establish the invalidity of an argument

Example 10: consider the argument below

$p \rightarrow q$
q
----
p

The truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Using a truth table to establish the invalidity of an argument

Example 10: consider the argument below

$p \rightarrow q$
q
----
p

The truth table:

p	q	$p \rightarrow q$
T	**T**	**T**
T	F	F
F	**T**	**T**
F	F	T

Shows that in one case when both hypotheses are true, the conclusion is false.

Hence *the argument is invalid*