Chapter 1. The Foundations: Logic and Proofs

Sections at zyBooks:

Propositional Logic

proposition – is a sentence that declares a fact, i.e. *declarative statement*, that is either *true* or *false*, but not both.

examples:

“It is raining now”

1+6 = 10

Washington, D.C. is the capital of United States of America

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examples:

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1+6 = 10

Washington, D.C. is the capital of United States of America

not propositions:

"What time is it now?"

x+5=10

"Please, stand clear of the closing doors."

Do you see the difference between the two sets of examples?
Why the last three sentences are not propositions?

Propositional Logic

proposition – is a sentence that declares a fact, i.e. *declarative statement*, that is either *true* or *false*, but not both.

we use letters to denote propositional variables (i.e. variables that represent propositions): *p, q, r, s*

examples:
p : 1+7=10
q : "It is sunny outside"

If a proposition is true, its value can be denoted by T or True or 1
If a proposition is false, its value can be denoted by F or ⊥ (bottom) or False or 0

The area of logic that deals with propositions is called the *propositional logic* or *propositional calculus*.
It was first developed by the Greek philosopher Aristotle.

Can we build/construct new propositions?

Propositional Logic

New propositions can be constructed from existing propositions using *logical operators*, and are called compound propositions.

logical operators:

			other denotations:	meaning:
¬	negation	¬p	$\overline{p}$, not p	"not p"
∧	conjunction	p ∧ q	p and q	"p and q"
∨	disjunction	p ∨ q	p or q	"p or q"

example 1:

Let proposition *p* stand for "I will go to a movie theater", then ¬*p* means "I will not go to a movie theater."

Truth table for negation operation:

p	¬p
T	F
F	T

or

p	¬p
1	0
0	1

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example 2:

Let proposition *p* stand for "It is raining" and *q* stand for "I want to go to a movie theater", then *p∧q* means "It is raining and I want to go to a movie theater."

Conjunction p ∧ q is true when both p and q are true. Truth table for p ∧ q:

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

or

p	q	p ∧ q
1	1	1
1	0	0
0	1	0
0	0	0

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∨	disjunction	p ∨ q	p or q	"p or q"

example 3:
Let proposition *p* stand for "It is raining" and *q* stand for "I want to go to a movie theater", then *p*∨*q* means "It is raining or I want to go to a movie theater."

Disjunction p ∨ q is true when at least one of p and q is true. Truth table for p ∨ q:

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

or

p	q	p ∨ q
1	1	1
1	0	1
0	1	1
0	0	0

Propositional Logic

more logical operators:

$\rightarrow$	implication	$p \rightarrow q$	" if p then q", "p implies q" *see page 6 for more*
$\oplus$	exclusive or	$p \oplus q$	"either p or q"

Implication:

$p \rightarrow q$

hypothesis, or antecedent, or premise (p)

conclusion, or consequence (q)

if p is true then q holds

Implication is also called conditional statement.

Truth table for the implication $p \rightarrow q$:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

example: Let p: "the weather is good", and q: "we'll go to the beach".

Then $p \rightarrow q$ stands for "If the weather is good we'll go to the beach"

Propositional Logic

more logical operators:

$\rightarrow$	implication	$p \rightarrow q$	" if p then q", "p implies q"
$\oplus$	exclusive or	$p \oplus q$	"either p or q"

<u>Exclusive or</u>:

$p \oplus q$ is true when exactly one of p and q is true

<u>example</u>: Let p: "the weather is good", and q: "we'll go to the beach".

Then $p \oplus q$ stands for "Either the weather is good or we'll go to the beach"

Truth table for the exclusive or $p \oplus q$:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Propositional Logic

even more logical operators:

$\leftrightarrow$ biconditional statement $p \leftrightarrow q$ " p if and only if q", or "p iff q"

Biconditional statement $p \leftrightarrow q$ is true when *p* and *q* have the same truth values, and is false otherwise.

example: Let p: "we'll go to the beach", and q: "the weather is good".

Then $p \leftrightarrow q$ stands for "we'll go to the beach if and only if the weather is good "

Truth table for $p \leftrightarrow q$:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional statements are also called bi-implications.

Propositional Logic

List of considered logical operators:

$\neg$	negation	$\neg p$	"not p"
$\wedge$	conjunction	$p \wedge q$	"p and q"
$\vee$	disjunction	$p \vee q$	"p or q"
$\rightarrow$	implication	$p \rightarrow q$	" if p then q", "p implies q"
$\oplus$	exclusive or	$p + q$	"either p or q"
$\leftrightarrow$	biconditional statement	$p \leftrightarrow q$	" p if and only if q"

Compound propositions and truth tables

Example: a truth table for compound proposition $(p \land q) \rightarrow (\neg r \lor q)$

p	q	r	$p \land q$	$\neg r$	$\neg r \lor q$	$(p \land q) \rightarrow (\neg r \lor q)$
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

Example: a truth table for compound proposition $(p \wedge q) \rightarrow (\neg r \vee q)$

p	q	r	$p \wedge q$	$\neg r$	$\neg r \vee q$	$(p \wedge q) \rightarrow (\neg r \vee q)$
T	T	T	**T**			
T	T	F	**T**			
T	F	T	**F**			
T	F	F	**F**			
F	T	T	**F**			
F	T	F	**F**			
F	F	T	**F**			
F	F	F	**F**			

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T	T	F	T	T		
T	F	T	F	F		
T	F	F	F	T		
F	T	T	F	F		
F	T	F	F	T		
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T	T	F	T	T	**T**	
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T	F	F	F	T	**T**	
F	T	T	F	F	**T**	
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T	F	T	F	F	F	T
T	F	F	F	T	T	T
F	T	T	F	F	T	T
F	T	F	F	T	T	T
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Propositional Logic

List of considered logical operators:

¬	negation	¬p	"not p"
∧	conjunction	p ∧ q	"p and q"
∨	disjunction	p ∨ q	"p or q"
→	implication	p → q	" if p then q", "p implies q"
⊕	exclusive or	p + q	"either p or q"
↔	biconditional statement	p ↔ q	" p if and only if q"

Let's play with English:

Let p and q be propositions; p: "Swimming at the New Jersey shore is prohibited" and q: "Sharks have been spotted near the shore".

Let's express the following propositions as English sentences.

a) $\neg p$

b) $\neg p \wedge \neg q$

c) $p \leftrightarrow q$

d) $\neg p \vee (p \wedge q)$

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Let's play with English:

Let p and q be propositions; p: "Swimming at the New Jersey shore is prohibited" and q: "Sharks have been spotted near the shore."

Let's express the following propositions as English sentences.

a) $\neg p$: Swimming at the New Jersey shore is allowed

b) $\neg p \wedge \neg q$: Swimming at the New Jersey shore is allowed and Sharks haven't been spotted near the shore

c) $p \leftrightarrow q$: Swimming at the New Jersey shore is prohibited if and only if Sharks have been spotted near the shore.

d) $\neg p \vee (p \wedge q)$ Swimming at the New Jersey shore is allowed or, it is prohibited and sharks have been spotted near the shore.

Converse, Contrapositive, and Inverse

Let's start with an implication (conditional statement) $p \to q$

$q \to p$	is the converse
$\neg q \to \neg p$	is the contrapositive
$\neg p \to \neg q$	is the inverse

! The contrapositive and the original statement have the same truth tables.

Example: what are the contrapositive, the converse and the inverse of the following conditional statement: “If you get 100% on the final, then you will get an A”?

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Example: what are the contrapositive, the converse and the inverse of the following conditional statement: "If you get 100% on the final, then you will get an A"?

(p = you get 100% on the final; q = you will get an A)

contrapositive: If you won't get an A then you didn't get 100% on the final
$\neg q \to \neg p$

converse: If you will get an A then you get 100% on the final
$q \to p$

inverse: If you don't get 100% on the final, then you won't get an A
$\neg p \to \neg q$

Propositional Logic

Precedence of Logical Operators:

$\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\oplus$

first *last*

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first *last*

Example:

$\neg p \vee q \rightarrow r \wedge p \equiv$

Propositional Logic

Precedence of Logical Operators:

$\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\oplus$

first … *last*

Example:

$\neg p \vee q \rightarrow r \wedge p \equiv ((\neg p) \vee q) \rightarrow (r \wedge p)$

Translating English Sentences

There are many reasons to translate English sentences into expressions involving propositional variables and logical connectives.

<u>One of them</u>: ambiguity of English (as well as any other language's). Translation into compound propositions removes the ambiguity.

<u>Another</u>: once we did the translation we can analyze these logical expressions to determine their truth values, manipulate the, use rules of inference (will be discussed later) to reason about them.

Translating English Sentences

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"You can go swimming if you know how to swim and the water is not too cold"

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therefore, let
p: "you can go swimming"
q: "you know how to swim"
r: "the water is (too) cold"   another option: r: "the water is not too cold"

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Then we get: $(q \wedge \neg r) \rightarrow p$

Example 2: translate a given sentence into a logical expression; determine whether an inclusive($\vee$) or exclusive($\oplus$) **or** is intended.

a) “To take discrete math, you must have taken calculus or a course in computer science”

b) “When you buy a new car from company A, you can get $2000 cash back or a 2% car loan”

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p: ”you can take discrete math”
q: ”you took calculus”
r: “you took a course in computer science”

$(q \vee r) \rightarrow p$

b) “When you buy a new car from company A, you can get $2000 cash back or a 2% car loan”

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p: "you can take discrete math"
q: "you took calculus"
r: "you took a course in computer science"
$(q \vee r) \rightarrow p$

b) "When you buy a new car from company A, you can get \$2000 cash back or a 2% car loan"

p: "you buy a new car from a company A"
q: "you can get \$2000 cash back"
r: "you can get a 2% car loan"
$p \rightarrow (q \oplus r)$ *most likely*

Note that during translation we may make a set of reasonable assumptions based on the intended meaning of the sentence.