More examples for Chapter 7.

8.2 Binomial coefficients and combinatorial identities

Example 1:
How many ways are there to select 6 cards from a standard deck of 52 cards?

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$$C(52,6) = \binom{52}{6} = \frac{52!}{6!(52-6)!} = \frac{52!}{6!(46)!} = \frac{47*48*49*50*51*52}{1*2*3*4*5*6} = 20{,}358{,}520$$

Answer: there are 20,358,520 ways to select 6 cards from a standard deck of 52 cards

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Bit (binary) strings contain *0*s and *1*s.

We need the bit strings that have six *1*s and four *0*s.

1 1 1 1 1 1 0 0 0 0 or 1 1 1 0 1 0 0 1 1 1
1st 2nd 3rd 4th 5th 6th 7th 8th 9th 10th 1st 2nd 3rd 4th 5th 6th 7th 8th 9th 10th

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1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}	7^{th}	8^{th}	9^{th}	10^{th}		1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}	7^{th}	8^{th}	9^{th}	10^{th}

Order of digits is important? Yes

If order of *0*'s important? No

Consider this then: we have 10 objects (1^{st}, 2^{nd}, …,10^{th}), 4 of them we choose to be zeros, the rest will be ones.

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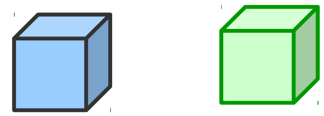
Consider this then: we have 10 objects (1^{st}, 2^{nd}, …,10^{th}), 4 of them we choose to be zeros, the rest will be ones.
Therefore, we are looking for 4-combination out of 10 objects.

$$C(10,4) = \binom{10}{4} = \frac{10!}{4!(12-4)!} = \frac{10!}{4!(6)!} = \frac{7\times8\times9\times10}{4\times3\times2\times1} = 210$$

Answer: there are 210 bit strings of length 10 that contain exactly 4 0's.

Example 3:

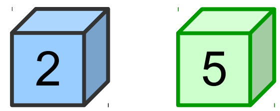
We roll two dice. How many outcomes are there in which we get at most 6?

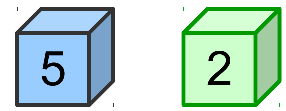


Solution: places:

possible outcomes: 1-6 1-6

Order of digits is important? Yes





1 1-5 / 2 1-4 / 3 1-3 / 4 1-2

1 x 5 + 1 x 4 + 1 x 3 + 1 x 2

5 1

+ 1 x 5 = 5 + 4+ 3 + 2 + 5 = 19

Answer: there are 19 outcomes that will give at most 6

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Use the complement!!!   At most one 1, i.e. zero 1's and one 1

picking out 0 elements from set of 8:

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All the possible strings: 2^8 = 256

256 – 1 – 7 = 248

Answer: there are 248 8-bit strings that contain at least two 1's

Recall r-combination or r-subset

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A simple combinatorial identity

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Let's check on numbers:

$$\binom{10}{4} = \frac{10!}{4!(10-4)!} = \frac{10!}{4!6!}$$

$$\binom{10}{6} = \frac{10!}{6!(10-6)!} = \frac{10!}{6!4!}$$

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$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = 4 \quad \binom{4}{4} = \frac{4!}{4!(4-4)!} = 1$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Binomial coefficients satisfy many different identities. We will see one of the most important of these.

[Pascal's Identity]

Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

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This identity is the basis for a geometric arrangement of binomial coefficients is a triangle, called Pascal's triangle.

Pascal's triangle

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1